Mrs. Anna Cabot (Jackson)

AN

# INTRODUCTION

TO

# GEOMETRY

AND THE

# SCIENCE OF FORM.

PREPARED FROM THE MOST

APPROVED PRUSSIAN TEXT-BOOKS.

STEREOTYPE EDITION, CAREFULLY REVISED.

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1846.

I HAVE carefully examined the manuscript of "An Introduction to Geometry" and think it admirably adapted to supply an important want in education. It is not a mere geometrical logic, but a natural and simple introduction to the Science of Form. By a beautiful and original series of inductive processes, it avoids tedious demonstrations, develops the taste for observation, which is so strong in the quick mind of youth, and leads the pupil to a real and practical knowledge of the truths of Geometry with a rapidity which would not have been anticipated. From these considerations, and from observing the strange neglect into which this science has fallen in our schools, I have strongly urged the publication of this excellent treatise, and think that its study should be insisted upon, as a valuable preliminary to a good education either at college or in business.

BENJAMIN PEIRCE, *Perkins Professor of Astronomy and Mathematics in Harvard University.*

Cambridge, April 21, 1843.

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# PREFACE.

Do my readers remember the fable of the contest between Nature and Education?—"Nature chose a vigorous young pine, the incipient mainmast of a man-of-war. But while she was feeding her pine with a plenty of wholesome juices, Education passed a strong rope round its top, and pulling it downwards with all her force, fastened it to the trunk of a neighboring oak. The pine labored to ascend, but not being able to surmount the obstacle, it pushed out to one side, and presently became bent like a bow. Still, such was its vigor, that its top, after descending as low as its branches, made a new shoot upwards; but its beauty and usefulness were quite destroyed."

Let the pine in this short tragedy represent those childish faculties which long to become acquainted with the actual world, and the cunning senses which wait on these faculties, and we have a true tale of the geometrical non-culture of the young.

The curiosity which speaks in children's busy eyes and hands should be to us the voice of Nature, bidding us make our beginnings early. The infant, who cannot speak, gazes earnestly and thoughtfully at the most common object, returning to it, and glancing from one part to another, as if to learn their connection. When he can walk, he goes round it, handling it, and studying it with all his senses. When he speaks, his questions are of size, form, and distance. If our answers are careless or unsatisfactory, his quick eyes and mind, not blunted by habit, detect our errors. He loves com-

parison of objects, and the imaginary multiplication and extension of them; he is pleased with the new and the different, and equally pleased with resemblance and equality to things known before.

Happy age of natural geometry; when each look and motion, nay his very games, lead the boy on to the laws which shape the spheres and hold the planets in their course. But we put a rope around the pine—and the eye grows dull and careless, and the faculties are starvelings—and the youth hears with a stupid indifference what he sought instinctively ten years before. He no longer loves comparison—his conceptive faculty disdains such materials; he cares not for properties of circles and triangles, things long familiar to his eye, and whose closer acquaintance he does not desire.

Feeling this injustice to nature, I sought a work on Geometry, which should connect the instincts of the child with the studies of the youth; which should in a pleasurable manner develop the powers, inform the memory, strengthen, sharpen, and expand the understanding. I found some German works which pleased me; particularly, one by Diesterweg, an author of distinguished reputation, and the Director of a Prussian Normal School, which treats the subject in a masterly manner, though it does not bring it down, as I wished, to the busiest and most questioning age.

From these works, at my suggestion, the "Introduction to Geometry" has been prepared. Many alterations and additions were necessary to adapt them to the modes of instruction usually pursued in our schools; but the original and natural mode of development has been carefully preserved, and must interest students of every age.

All is tangible, orderly, and leading to some conclusion. Solids are first presented, and their properties, particularly symmetry of form, displayed; different solids are compared, and each is minutely described by the pupil, and a skeleton of its surface constructed. He considers the solid as bounded by surfaces; the surfaces by lines; and at last he separates the immaterial form from the material substance.

Then lines are taken up, and the possible number and variety of their intersections are proved geometrically by interesting exercises, and arithmetically by series which might be difficult, were they not presented in so simple and orderly a manner. Angles are treated in the same systematic manner, and experiments are made on them, requiring much ingenuity and accuracy.

Then follow the principal propositions of the Elements of Geometry, which are made more interesting to the student by introducing examples of their practical utility. The book thus recommends itself equally to the practical student, and to him to whom it is the stepping stone to more abstruse knowledge; and it is hoped that it will contribute to make this study, which is nothing less than the study of the laws according to which the universe exists, as popular as it should be.

THE AUTHOR OF THE THEORY OF TEACHING.

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## EXPLANATION OF SIGNS.

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MATHEMATICIANS have adopted the following Signs to express the operations and relations which are of most frequent occurrence.

$+$ is the Sign of addition. For example, AB $+$ BC means that the length of the line AB is to be added to that of BC. It is read AB *plus* (or *more*) BC. The result of the operation is called the *sum* of the two lines.

$-$ is the Sign of subtraction. AB $-$ CD means that the length of CD is to be taken from AB. It is read AB *minus* (or *less*) CD. The result of the operation is the *difference* of the two lines.

$\times$ is the Sign of multiplication. AB $\times$ CD is read AB *into* CD, and the result of the operation is the *product*.

$\div$ is the Sign of division. AB $\div$ CD is read AB *divided by* CD, and the result of the operation is the *quotient*.

$>$ is the Sign for *greater than*. AB $>$ CD means that the line AB is longer than CD.

$<$ is the Sign for *less than*. AB $<$ CD means that the line AB is shorter than CD.

$=$ is the Sign of equality. AB $=$ CD means that the line AB is equal to the line CD. The whole forms an *equation*.

2*

$\backsim$ is the sign of similarity. ABC $\backsim$ DEF means that the triangle ABC is similar to DEF.

$\overline{AB}^2$ means a square each side of which is equal to the line AB.

To avoid the frequent repetition of the term *right angle*, the abbreviation R. A. is substituted. Thus 2 R. A. is read *two right angles*.

# PART FIRST.

## EXAMINATION AND IMITATION.

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IN the following exercises the real solid bodies should be presented to the scholars. The scholars should first examine the solid which is the subject of the exercise, and then solve the questions and problems which may be proposed. It is intended that the drawings should be made without the assistance of a ruler, or of mathematical instruments.

### THE CUBE.

**1.** *What is to be remarked in the Cube?*

In the cube we remark 6 surfaces,—1 upon which it rests, 1 opposite to-that, and 4 upright surfaces; 12 straight lines which bound the surfaces,—4 above, 4 below, and 4 upright; 24 angles made by the meeting of these straight lines, 4 in each surface; 8 corners made by the meeting of 3 surfaces,—4 above and 4 below; 12 angles made by the meeting of 2 surfaces,—1 at each straight line; 3 surface axes, that is, imaginary straight lines from the middle of one surface to the middle of the opposite surface, upon which the cube may be supposed to turn; 4 corner axes, or imaginary straight lines pass-

ing from corner to corner, through the middle point of the cube; 6 line axes, or imaginary straight lines passing through the cube as before and joining the middle points of two lines.

Each of these surfaces is a *plane* surface, or simply a *plane;* that is, if we take two points in any part of it and connect them by a straight line, such line will touch the surface through its whole length. The surfaces of the cube, and indeed of every solid bounded by planes, are called the *faces* of the solid; the lines where the faces meet are called *edges* or *sides.* The angles made by the meeting of two faces are called *plane* angles; the corners, or angles made by the meeting of 3 faces, are called *solid* angles.

**2.** Let us seek if any of the parts which have been mentioned are equal one to another. Each corner is of the same size and shape as each of the others. All 6 faces are equal one to another. No side is longer or shorter than another side; all are equally long. All the 24 line angles are equal. Each face has 4 equal angles.

A figure having 4 angles is called a *tetragon,* from two Greek words, meaning *four* and *angle;* or more commonly a *quadrilateral,* which means *four-sided,* because it has also 4 sides. Every quadrilateral in which all the sides and angles are respectively equal, one to another, is called a *square;* thus, the faces of the cube are squares. Every angle of the square is a *right* angle; that is, such an angle as is made by one straight line meeting another so as to make the *adjacent* angles, (that is, the angles made on each side of itself,) equal; such lines are *perpendicular* one to the other. Thus, the adjacent sides of the cube are perpendicular to one another, because they make right angles with each other. The

faces also make right angles with one another, and are therefore perpendicular.

The opposite faces of the cube are *parallel* to each other; that is, they are in all parts at the same distance from each other, and would never meet however far they were extended in every direction. If the cube lies upon one of its faces upon a table, the surface of which is parallel to water at rest, the position of such face and of its bounding lines is *horizontal;* the position of the opposite face and its bounding lines is also horizontal; the faces have like positions, they are parallel. The other faces of the cube have an upright or *vertical* position; that is, such a position as a leaden weight, hanging freely, gives to a string.

QUESTIONS.

1. What relative positions have the faces of the cube? Which are opposite to one another?

2. To how many edges is each edge of the cube perpendicular?

3. To how many faces of the cube is each edge perpendicular?

4. To how many faces is each face perpendicular?

5. How many faces of the cube are parallel one to another? 3

6. If the cube lies on a horizontal surface, how many of the edges will have a horizontal, and how many a vertical position?

**3.** You may now draw upon your slate all that has been remarked in the cube.

1. *Straight lines.* Draw straight lines from above to

below, from below to above, from the left to the right, from the right to the left, from left below to right above, from right below to left above, from left above to right below, from right above to left below. Draw, in each of the foregoing 8 directions, 6 straight lines parallel to one another, and of equal length.

Draw 2 straight lines which shall touch at one point.

Draw many pairs of lines which shall come together, if *produced*, that is, made longer.

Draw many lines, which, if produced, will form one line.

2. *Angles.* Draw two straight lines which shall be perpendicular to one another, and thus make a right angle.

Make right angles in all the positions in which you have drawn the first straight lines.

Point out the lines on the walls or furniture of this room which appear to form right angles.

3. *Faces.* Draw lines forming a square.

Draw many squares in different positions.

Draw 6 squares of which each succeeding one shall be larger than the one preceding it.

Draw many pairs of squares, in each of which one square shall be equal to the other.

## THE TRIANGULAR PRISM.

**4.** We will now examine this upright *triangular* prism. It is bounded by 5 faces; 2 of these are *triangles*, that is, figures having 3 angles; and 3 are quadrilaterals. The 2 triangles are placed opposite one to the other, are parallel, and of equal size, and their sides

are of equal length; they are called the *bases* of the prism. From the form of the bases the prism takes its distinctive name; thus the cube is a *quadrangular* prism. In the triangular prism the 3 quadrilaterals touch one another; each joins the other two; together they form the *convex surface* of the prism. Upon the triangles it is said to *stand;* upon the quadrilaterals to *lie.*

The triangular prism has 6 corners, or *solid angles.* If the prism *stands*, 3 corners are placed above and 3 below; if it *lies*, 4 corners will be below, 2 above. At each corner 3 line angles meet.

Each triangle is bounded by 3 *edges* or *sides;* each quadrilateral by 4. There are 9 sides; each side belongs to two figures; 6 of them, each to one triangle and one quadrilateral; the other 3, each to two quadrilaterals. At each corner 3 sides meet. In this prism the 3 sides belonging solely to the quadrilaterals are perpendicular to those which belong also to the triangles; the sides of the triangles are not perpendicular one to another. The opposite sides of the quadrilaterals are parallel each to each. The 6 sides common to the triangles and quadrilaterals are equal one to another; the 3 sides which belong to the quadrilaterals alone are likewise equal one to another.

**5.** We will now consider the angles.

1. *The plane angles.* At each side there is a plane angle. The triangular prism has 9 plane angles; 6 at the sides common to the triangles and quadrilaterals, and 3 at those which belong to the quadrilaterals alone. In this prism the former are right angles, the latter are not right angles.

2. *The line angles.* Each triangle has 3 line angles, each quadrilateral has 4. Thus in the 2 triangles there

are 6; in the 3 quadrilaterals there are 12; consequently in the whole prism there are $6 + 12 = 18$ line angles. Three of these angles meet at each corner. The 12 angles of the quadrilaterals are right angles. The 6 angles of the triangles are not right angles. The corresponding angles of the two triangles in all prisms must have a like position; and the lines which form the corresponding angles are respectively parallel.

In the triangles each side is opposite to an angle; in the quadrilaterals each side is opposite to another side, and each angle to another angle.

In the triangular prism we can suppose 1 surface axis; 3 edge-face axes; 6 corner-edge axes.

We will now examine triangular prisms which are not upright, those whose bases have unequal sides, and those which are truncated so that their bases are no longer parallel to one another.

**6.** Angles which are not right angles have the common name of *oblique* angles. An oblique angle may be greater or less than a right angle; if it be greater than 1, but less than 2 right angles, it is called an *obtuse* angle; if it is greater than 0 and less that 1 right angle, it is called an *acute* angle. We compare these angles with a right angle, because the magnitude of a right angle is constant, and always remains equal to itself. It follows from the definition of a right angle, that all right angles are equal. But all obtuse angles are not equal; neither are all acute angles equal.

The magnitude of an angle does not depend upon the length of the lines which form it, and which are called the *sides* or sometimes the *legs* of the angle; but upon their inclination one to the other. We may make the sides longer or shorter, still the magnitude of the angle re-

mains the same; but change the mutual inclination of the sides, and the magnitude of the angle is changed.

**7.** A quadrilateral in which the sides opposite to each other are parallel, is called a *parallelogram*.

Draw upon your slates the following figures:—

1. A right angle; an obtuse angle; an acute angle.
2. 6 acute angles; of which each succeeding shall be greater than that which precedes it.
3. 6 obtuse angles; of which each succeeding shall be greater than the one which precedes it.
4. A parallelogram.
5. 2 parallelograms; one with right angles, and the other with oblique angles.
6. Parallelograms in which two of the sides shall be twice, thrice, 4 times longer than the other two.
7. A parallelogram having its sides equal and its angles equal; that is, *a square.*
8. A parallelogram in which all the angles shall be right angles, but the sides shall not be equal; that is, a *rectangle.*
9. A parallelogram with equal sides, but unequal angles, that is, a *rhombus* or *lozenge;* then a parallelogram with unequal sides and unequal angles; that is, a *rhomboid.* You will remark that in the rhomboid, as also in the rectangle, the sides opposite to each other are equal.
10. A quadrilateral having two sides parallel to each other, the other two not parallel; that is, a *trapezoid;* then a quadrilateral having no two sides parallel, that is, a *trapezium.*
11. A triangle with one right angle; called a *right angled* triangle, or simply a *right triangle.*

A triangle with one obtuse angle; called an *obtuse angled* triangle.

A triangle with three acute angles; called an *acute angled* triangle.

A triangle with three equal sides; called an *equilateral* triangle.

A triangle with two of its sides equal; called an *isosceles* triangle.

A triangle with no two sides equal; called a *scalene* triangle.

*Note.*—Other prisms, as the pentagonal, hexagonal, &c., may be examined in a similar manner.

**8.** A figure bounded by straight lines is called a *rectilineal* figure or *polygon*. A polygon is *regular* if all its sides are equal, and all its angles are equal; otherwise it is *irregular*.

*Curvilineal* figures are bounded by curved lines; and *mixtilineal* partly by curved, and partly by straight lines.

**9.** Draw upon your slates the following figures:—

A triangle with equal sides and equal angles, that is, a *regular* triangle.

A regular quadrilateral; a regular 5-sided figure, that is, a *pentagon*.

A regular 6-sided figure, that is, a *hexagon*.

A regular 7-sided figure, that is, a *heptagon*.

A regular 8-sided figure, that is, an *octagon*.

A regular 9-sided figure, that is, a *nonagon*.

A regular 10-sided figure, that is, a *decagon*.

An irregular triangle, quadrilateral, pentagon, &c., to the decagon.

### THE CYLINDER.

**10.** Let us now examine the cylinder. A round pillar is a cylinder *standing up;* a roller is a cylinder *lying*

*down.* The cylinder is bounded by 2 *plane* surfaces, and 1 *curved* surface. The 2 plane surfaces, called the *bases* of the cylinder, are parallel; they are of equal size, and each is bounded by a curved line, all parts of which are equally distant from the middle point of the surface. In this upright cylinder the bases are perpendicular to the curved surface, or *convex* surface of the cylinder, as the whole taken together is called. The convex surface is curved in one direction; in the direction from one base to the other straight lines may be drawn in it. We can suppose a straight line joining the middle of the bases; this is the *axis* of the cylinder.

If we suppose a rectangle to revolve about one of its sides as an axis, the side opposite to this axis will describe the convex surface of the upright cylinder, and the other two sides will describe its bases.

We will now examine the oblique cylinder, the fluted cylinder, and the cylinder which has been cut in a direction not parallel to the base.

**11.** A surface no part of which is plane is called a *curved* surface. A plane surface bounded by a curved line, all parts of which are equally distant from the middle point, or *centre* of the plane, is called a *circle.* The curved line bounding it is called a *circumference.* The term *circle* is sometimes improperly used for *circumference;* but a circle is a surface; a circumference is a line bounding such surface. Any portion of the circumference is called an *arc*, from a Latin word meaning a *bow;* a straight line joining the two extremities of an arc is called a *chord*, from a Latin word meaning a *string.* A chord divides the circle and its circumference into two parts; each portion of the circle is called a *segment;* each portion of the circumference is an *arc;* thus a segment

is a surface; an arc is a line. If the chord passes through the centre of the circle it is called a *diameter*, and it will divide the circle and its circumference into two equal parts, called respectively *semi-circles* and *semi-circumferences*. A straight line drawn from the centre to any point in the circumference is called a *radius*. Two radii drawn in directly opposite directions form a diameter. A straight line which, however far it may be produced in both directions, touches the circumference only at one point, is called a *tangent:* the point where it touches the circumference is called the *point of contact.*

**12.** Draw upon your slates these lines and figures:—

A circumference; mark the centre of the circle. Draw a radius, a diameter, another chord, and a tangent.

Six circumferences of circles having a common centre.

Two circumferences intersecting each other.

Divide a circle and its circumference into 4 equal parts by diameters.

An equilateral triangle in a circle, so that the sides of the triangle shall be chords of the circle.

An equilateral triangle about a circle, so that the sides of the triangle shall be tangents to the circle.

A square in a circle. A circle about a square.

A square about a circle. A circle in a square.

A regular pentagon in a circle. A circle about a regular pentagon.

A regular pentagon about a circle. A circle in a regular pentagon.

*Note.*—The *pyramid*, the *cone*, and the *sphere* can be examined and treated in a manner similar to that in which we have treated the prisms and the cylinder.

**13.** Let us now compare together the cube, the triangular prism, and the pentagonal prism. In order to avoid the frequent repetition of the names, we will designate the triangular prism by the letter A, the cube by B, and the pentagonal prism by C.

1. *Surfaces.* All three bodies are bounded by planes; A by 5; B by 6; and C by 7. The convex surface of each is composed of rectangles; A has 3; B has 6; C has 5. A is likewise bounded by 2 triangles; C by 2 pentagons.

2. *Corners* or *solid angles.* A has 6; B has 8; C has 10.

3. *Edges* or *sides.* A has 9; B has 12; C has 15. A has 6 sides equal one to the other, and 3 equal one to the other; B has all the sides equal. C has 10 equal one to another, and 5 equal one to another.

4. *Plane angles.* A has 9; B has 12; C has 15. In A 6 are right angles, and 3 acute angles. In B there are 12 right angles. In C are 10 right angles, and 5 obtuse angles.

5. *Line angles.* A has 18; B has 24; C has 30 line angles.

In A 6 angles are acute, and 12 are right angles; in B all the 24 angles are right angles; in C 20 are right angles, and 10 are obtuse angles. At each corner of A are 2 right and 1 acute angles; at each corner of B are 3 right angles; at each corner of C are 2 right and 1 obtuse angles: thus we have 2 right angles at each corner of each of these bodies.

6. *Axes.*

| | | | | |
|---|---|---|---|---|
| Face axes | A has 1. | B 3. | C 1. |
| Corner axes | A 0. | B 4. | C 0. |
| Edge axes | A 0. | B 6. | C 0. |
| Corner face axes | A 0. | B 0. | C 0. |
| Corner edge axes | A 6. | B 0. | C 10. |
| Edge face axes | A 3. | B 0. | C 5. |

## DRAWING THE OUTLINES OF THE SOLID BODIES.

**14.** We will now endeavor to represent upon a plane surface the 5 faces of the triangular prism in connexion. This body is bounded by 2 equilateral triangles opposite one to the other, and 3 rectangles in contact one with another; and our drawing, or *diagram*, must conform to this. We must therefore construct 3 rectangles in contact, and the 2 triangles, one at each end of one of the rectangles. (Fig. 1.)

In a similar manner may be represented the quadrangular, pentagonal, and hexagonal prism; the cube; the cylinder; the cone; the triangular, quadrangular, pentagonal, and hexagonal pyramid. (Figures 2 to 11.)

*Note.*—It would be useful to have these representations of the solid bodies drawn on pasteboard, the figure cut out all round, and incisions made in the lines where the several parts of the bodies unite, so that the pasteboard may bend and form the bodies drawn upon it. By means of these the truth of the representations will at once be perceived.

## THE REGULAR SOLIDS.

**15.** A *regular solid* is one, all the faces of which are equal regular polygons; that is, in which all the line, plane, and solid angles, all the faces, and all the sides, are respectively equal. There are five regular solids, the names of which are formed from the Greek word signifying *seat*, or *face*, combined with the Greek numeral denoting the number of seats or faces in the solid.

1. The *tetraedron*, which is bounded by 4 equilateral triangles.

2. The *octaedron*, bounded by 8 equilateral triangles.

3. The *icosaedron*, bounded by 20 equilateral triangles.

4. The *hexaedron*, or *cube*, bounded by 6 squares.

5. The *dodecaedron*, bounded by 12 regular pentagons.

The name *polyedron* is used to denote any solid bounded by planes, whether it be regular or irregular. It is derived from the Greek, and signifies *many-faced.*

Exercises, similar to those which have preceded, may be performed with the regular solids. It will be very useful to have models of these solids, made either of wood or pasteboard. The scholars may make these models for themselves, of clay, turnips, potatoes, &c. It will be very useful for them to draw upon paper, or some plane surface, a *skeleton*, or representation of the faces of each, as they are shown in figures 5, 77, 78, 79, 80.

## EXERCISES ON FIGURES.

**16.** The teacher may dictate to the scholars a figure, that is, he may tell them what lines to draw upon their slates; or he may himself draw a figure upon the board, and desire the scholars to tell, or draw, all that they remark in the figure. A few examples will suffice to show the mode of proceeding.

Write down all that you remark in this figure, (having drawn figure 12.) The scholars will find as follows.

*Figures.* 1 square; 4 triangles,—each composed of 2 triangles; 4 smaller triangles; 4 pentagons.

*Lines.* 6 lines,—4 which bound the square, and 2 inside the square; 3 lines meet at each corner; the two interior lines cut each other at one point.

*Angles.* 4 angles in the square; 4 at the point in the middle; the 4 angles of the square are divided into 8

parts; in the 4 large triangles are 12 angles; and in the 4 small triangles there are likewise 12 angles; 4 right angles in the square; 4 right angles at the middle point; 8 acute angles in the triangles.

**17.** The bounding lines of the square, or of any polygon, taken together, form the *perimeter* of such polygon. The straight lines within the square are called *diagonals;* thus a *diagonal* is a straight line joining the vertices of any two angles of a polygon, which are not already connected by one of the bounding lines.

**18.** Tell me all that you remark in this figure. (Fig. 13.)

*Figures.* *Rectilineal.* The same as in fig. 12.
*Curvilineal.* 1 circle.
*Mixtilineal.* 4 large and 4 small two-angled figures; 4 large and 4 small triangles.

*Lines.* *Straight.* The same as in fig. 12.
*Curved.* 1 circumference; 4 semi-circumferences; 4 quarter-circumferences; 4 three-quarter-circumferences.

*Angles.* *Rectilineal.* The same as fig. 12.
*Mixtilineal.* 8 mixtilineal acute angles; 8 mixtilineal right angles; 8 mixtilineal obtuse angles.

**19.** We will now draw a figure from dictation; each one upon his slate.

A straight line; *bisect* it, that is, divide it into 2 equal parts; at the division point draw a line perpendicular to the first line, and of the same length; bisect the perpendicular; connect the end and middle points of the

perpendicular by straight lines with the end and middle points of the first line.

Now write down all that has been done; mention the figures that are formed; and whatever you remark in those figures.

**20.** In the preceding exercises we have become acquainted with:—

1. A *Solid*, which has extension in length, breadth, and thickness.

2. A *Surface*, which is the boundary of a solid, having extension in length and breadth only.

3. A *Line*, which is the boundary of a surface, and has extension only in length.

4. A *Point*, which is the extremity of a line, and has only a position; it has no extension, either in length, breadth, or thickness.

The science which treats of the measure of extension is called *Geometry*, from two Greek words which signify *to measure land;* thus denoting the purpose to which the science was first applied.

# PART SECOND.

## RECKONING AND CONSTRUCTION.

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### I. Points.

**21.** Let us consider *how often* 2, 3, 4, *or more points can be placed in a different order of succession.* We will first take

2 *points*, which we will designate by the letters $a$ and $b$.

We may have $a$ on the left, $b$ on the right;
$a$ on the right, $b$ on the left.

Thus we have 2 ways, $a\,b$, $b\,a$, that is, $1 \times 2$.

3 *points*, $a$, $b$, $c$. Two points give 2 ways, $a\,b$, $b\,a$. Now take $c$ with $a\,b$. It may occupy the 3d, 2d, or 1st place. This gives $a\,b\,c$, $a\,c\,b$, $c\,a\,b$, 3 ways. Now take $c$ with $b\,a$, and we shall again have 3 ways, $b\,a\,c$, $b\,c\,a$, $c\,b\,a$,—in all 6 ways, that is, $1 \times 2 \times 3$.

4 *points*, $a$, $b$, $c$, $d$. We can take the six preceding ways, and add the 4th point $d$ to each of the 6. The 1st was $a\,b\,c$; add $d$; it may occupy either the 4th, 3d, 2d, or 1st place; thus, $a\,b\,c\,d$, $a\,b\,d\,c$, $a\,d\,b\,c$, $d\,a\,b\,c$,

in all 4 ways. Just so each of the above 6 gives 4 ways, therefore in all $4 \times 6 = 24 = 1 \times 2 \times 3 \times 4$.

By observing the several series of numbers, which we have found, we may determine the general rule. On the addition of each successive point, the number of the preceding ways has been multiplied by the number which denotes the present number of points. Therefore to find the number of different combinations which can be made with any number of points; *Multiply together the natural series of numbers from one up to, and including, that number which denotes the number of the given points.*

**22.** *What is the greatest number of points at which any given number of straight lines may intersect one another?*

*Answer.* 2 *straight lines* can only touch each other at one point, and this point is common to the 2 lines.

3 *straight lines.* The 3d line may intersect each of the preceding; and thus we have 2 points more; this gives in the whole $1 + 2 = 3$ points.

4 *straight lines.* The 4th line may intersect each of the other 3, and thus we have 3 new points; $1 + 2 + 3 = 6$ points, (fig. 14.)

In a similar manner the 5th line may intersect each of the other 4 lines, and in general each successive line may intersect all which precede it. Thus with each successive line there will be as many new points as there were lines already.

| | | | | | |
|---|---|---|---|---|---|
| The 6th | line | gives | 5 | new | points. |
| 7th | " | " | 6 | " | " |
| 100th | " | " | 99 | " | " |

Rule. *To find the greatest number of points of intersection, which a given number of straight lines may make: add together the natural series of numbers from* 1 *up to, but without including, that number which makes the number of lines.*

| | | |
|---|---|---|
| 1 line gives | | 0 points. |
| 2 lines give | | 1 " |
| 3 " " | $1+2$ | $=$ 3 " |
| 20 " " | $1+2+3+\ldots\ldots+19$ | $=$ 190 " |

We can arrive at the same result by a shorter process. It is evident that straight lines may be drawn in such a manner, that each line shall intersect all the others. Take, for example, 20 lines; each of these may intersect the other 19. Thus there will be 19 points in each line; this would give in all the lines $19\times20=380$: but each point is common to two lines; we must therefore divide this product by 2; thus for 20 lines we have $\frac{20\times19}{2}=190$. Thus we get another rule; *Multiply the number of lines by the same number less* 1, *and divide this product by* 2.

**23.** We will now reverse the operation. *Let us suppose the greatest number of points at which a certain number of straight lines can intersect one another to be* 45; *what is the number of lines?*

*Answer.* We have already learnt that the number of intersection points is the half of the product of two numbers differing from each other only by 1. We must therefore multiply the given number by 2, and then seek for two numbers, differing from each other only by 1, which multiplied together will give this product. For

example, $45 \times 2 = 90 = 10 \times 9$; therefore the number of lines which will give 45 points of intersection is 10.

**24.** The intersecting straight lines may be divided into many sets.

We will first suppose them to be divided into 2 sets, and will consider several different cases.

1. *The lines of each set parallel among themselves.*

*Example.* Six lines are divided into 2 sets, one of which has 4 lines and the other 2 lines; the 4 are parallel one to the other, and the 2 are parallel to each other. What is the greatest number of intersection points? *Answer.* Each line of one set intersects each line of the other set, thus we have $4 \times 2 = 8$.

Ten straight lines may be divided into 2 sets as follows:—

| | | | | |
|---|---|---|---|---|
| 9 and 1, | which | gives | $9 \times 1 = 9$ | points. |
| 8 and 2, | " | " | $8 \times 2 = 16$ | " |
| 7 and 3, | " | " | $7 \times 3 = 21$ | " |
| 6 and 4, | " | " | $6 \times 4 = 24$ | " |
| 5 and 5, | " | " | $5 \times 5 = 25$ | " |

Observe the series of numbers, 9, 16, 21, 24, 25; the differences between the successive numbers form the series 7, 5, 3, 1.

2. *The lines of each set intersecting among themselves at one point, as in this figure.* (Fig. 15.) Here the lines of each set give one intersection point among themselves. The lines of one set also intersect the lines of the other. Multiply the number of lines in one set by the number of lines in the other set, and add 2 to the product; we shall thus have the number of intersection points.

The number 12 being divided into

| | | |
|---|---|---|
| 10 and 2 | gives | $(10 \times 2) + 2 = 22$ points. |
| 9 and 3 | " | $(9 \times 3) + 2 = 29$ " |
| 8 and 4 | " | $(8 \times 4) + 2 = 34$ " |
| 7 and 5 | " | $(7 \times 5) + 2 = 37$ " |
| 6 and 6 | " | $(6 \times 6) + 2 = 38$ " |

3. *The lines of one set parallel to one another, and the lines of the other set intersecting one another at one point.* In this case each line of the second set intersects each line of the first set, and the lines of the second set give one intersection point besides. To ascertain the whole number of points, we must multiply the numbers of the two sets together and add 1 to the product.

*Example.* Divide the number 16 into 2 sets as follows, the 1st column being the number of lines in the set of parallels.

| | | |
|---|---|---|
| 14 and 2 | give | $(14 \times 2) + 1 = 29$ |
| 13 and 3 | " | $(13 \times 3) + 1 = 40$ |
| 12 and 4 | " | $(12 \times 4) + 1 = 49$ |
| 11 and 5 | " | $(11 \times 5) + 1 = 56$ |
| 10 and 6 | " | $(10 \times 6) + 1 = 61$ |
| 9 and 7 | " | $(9 \times 7) + 1 = 64$ |
| 8 and 8 | " | $(8 \times 8) + 1 = 65$ |

The series of numbers, 29, 40, 49, 56, 61, 64, 65, gives the series of differences 11, 9, 7, 5, 3, 1.

4. *The lines of one set parallel, and the lines of the other set intersecting one another at the greatest number of points.* Each line of the second set intersects each line of the first set; the number of points at which the lines of the 2d set intersect one another must be calculated by the rule before given, (**22.**) The number 20, for example, being divided

Into 18 and 2, gives $(18 \times 2) + \frac{2 \times 1}{2} = 37$

17 and 3, " $(17 \times 3) + \frac{3 \times 2}{2} = 54$

16 and 4, " $(16 \times 4) + \frac{4 \times 3}{2} = 70$

15 and 5, " $(15 \times 5) + \frac{5 \times 4}{2} = 85$

14 and 6, " $(14 \times 6) + \frac{6 \times 5}{2} = 99$

13 and 7, " $(13 \times 7) + \frac{7 \times 6}{2} = 112$

12 and 8, " $(12 \times 8) + \frac{8 \times 7}{2} = 124$

11 and 9, " $(11 \times 9) + \frac{9 \times 8}{2} = 135$

10 and 10, " $(10 \times 10) + \frac{10 \times 9}{2} = 145$

The series of numbers, 37, 54, 70, 85, 99, 112, 124, 135, 145, gives a series of differences 17, 16, 15, 14, 13, 12, 11, 10.

By continuing the preceding calculations, (1, 2, 3,) a series of differences is obtained, the reverse of those already obtained.

By continuing the calculation for lines divided into sets in this manner, (4, 5,) the series simply progresses.

5. *The lines of one set intersecting one another at the greatest number of points, and the lines of the other set intersecting one another at one point.*

In this case, first calculate the number of points in the first set; then multiply the number of lines of the 2d set by the number of lines in the 1st set, because each line of the one set intersects each line of the other set, and add 1 to the product, because the lines of the 2d set also

intersect one another at one point. Divide the number 8

into 6 and 2, it gives $\frac{6\times 5}{2}+(6\times 2)+1=28$

into 5 and 3, " $\frac{5\times 4}{2}+(5\times 3)+1=26$

into 4 and 4, " $\frac{4\times 3}{2}+(4\times 4)+1=23$

**25.** Let us now suppose the lines to be divided into 3 sets.

1. *The lines in each set parallel among themselves.* In this case each line of one set can intersect each line of the other two sets. To find the greatest number of intersection points, the numbers which the sets contain must be multiplied together in pairs. If then the numbers in the 3 sets be

6, 2, and 2, we have $(6\times2)+(6\times2)+(2\times2)=28$ points
5, 3, and 2, " $(5\times3)+(5\times2)+(3\times2)=31$ "
4, 3, and 3, " $(4\times3)+(4\times3)+(3\times3)=33$ "
4, 4, and 2, " $(4\times4)+(4\times2)+(4\times2)=32$ "

2. *The lines of the* 1*st and* 2*d sets parallel among themselves, and the lines of the* 3*d set intersecting one another at one point.* In this case, the lines of the first two sets intersect one another, and the lines of the 3d set, besides intersecting all the lines of the other two, give one intersection point among themselves. If the numbers in the 3 sets be

8, 2, 2 we have $(8\times2)+[(8+2)\times2]+1=37$ points
7, 3, 2 " $(7\times3)+[(7+3)\times2]+1=42$ "
7, 2, 3 " $(7\times2)+[(7+2)\times3]+1=42$ "
6, 4, 2 " $(6\times4)+[(6+4)\times2]+1=45$ "
6, 2, 4 " $(6\times2)+[(6+2)\times4]+1=45$ "
6, 3, 3 " $(6\times3)+[(6+3)\times3]+1=46$ "

3. *The lines of each set intersecting among themselves at* 1 *point.* In this case every line of each set intersects every line of the other 2 sets, and there are 3 points besides. Therefore the numbers in the 3 sets must be multiplied together in pairs, and 3 be added to the product.

Suppose the number in the 3 sets to be

3, 3, 3 we have $(3 \times 3) + (3 \times 3) + (3 \times 3) + 3 = 30$
4, 4, 4 " $(4 \times 4) + (4 \times 4) + (4 \times 4) + 3 = 51$
6, 6, 6 " $(6 \times 6) + (6 \times 6) + (6 \times 6) + 3 = 111$
2, 3, 4 " $(2 \times 3) + (2 \times 4) + (3 \times 4) + 3 = 29$
3, 4, 5 " $(3 \times 4) + (3 \times 5) + (4 \times 5) + 3 = 50$

4. *The lines of the* 1*st set parallel to one another: those of the* 2*d set intersecting one another at* 1 *point: those of the* 3*d set intersecting one another at the greatest number of points.* The calculation must be made accordingly. If the 3 sets are

5,5,5 we have $(5 \times 5) + 1 + [(5+5) \times 5] + \left(\frac{5 \times 4}{2}\right) = 86$ ps.

6,6,6 " $(6 \times 6) + 1 + [(6+6) \times 6] + \left(\frac{6 \times 5}{2}\right) = 124$ "

8,8,8 " $(8 \times 8) + 1 + [(8+8) \times 8] + \left(\frac{8 \times 7}{2}\right) = 221$ "

6,10,4 " $(10 \times 6) + 1 + [(6+10) \times 4] + \left(\frac{4 \times 3}{2}\right) = 131$ "

## I. Lines.

**26.** Let us now calculate how many straight lines may be drawn between a given number of points, of which only two lie in the same direction.

1. *Between* 2 *points*, only 1 straight line can be drawn, and these determine the length and position of the line; a straight line, drawn from the 1st point to the

4*

2d, will coincide with a straight line drawn from the 2d to the 1st. To *coincide* is to fall on, and exactly fill the same space.

2. *Between* 3 *points.* In this case we may have 3 distinct straight lines, viz., from the 1st point to the 2d, from the 2d to the 3d, from the 3d to the 1st. We can state it in this manner; from each of the 3 points straight lines may be drawn to the other 2 points; thus, from each point 2 lines; consequently, from the 3 points $3 \times 2 = 6$ lines. But each line is thus counted twice; for the straight line from the 1st point to the 2d is the same as that drawn from the 2d to the 1st; therefore the product $3 \times 2$ must be divided by 2. The number of straight lines that can be drawn between 3 points is $\frac{3 \times 2}{2} = 3$.

3. *Between* 4 *Points.* Straight lines may be drawn from each of these 4 points to each of the other points; thus from each point 3 straight lines, and therefore from 4 points $4 \times 3 = 12$ straight lines. For the reason before given, this is twice the real number; the true number is $\frac{4 \times 3}{2} = 6$.

4. *Between five or more points.* We may take any number of points, the process will be the same. Straight lines can be drawn from each point to every other point. Therefore from each point as many lines can be drawn as there are points less 1. We can now form the general rule. *Multiply the whole number of points by the same number less* 1, *and divide the product by* 2.

Between 7 points may be $\frac{7 \times (7-1)}{2} = 21$ lines.

" 12 " $\frac{12 \times (12-1)}{2} = 66$ "

" 50 " $\frac{50 \times (50-1)}{2} = 1225$ "

The converse problem should be solved, namely, the number of straight lines which may be drawn between an unknown number of points being given, required the number of points.

**27.** Every part of a straight line which is bounded by two points in that line, is called an *extent.* Two or more contiguous extents form together a *compound extent.*

We will now calculate the number of single *extents* between a given number of points in a straight line.

| | | | |
|---|---|---|---|
| Between 2 | points | lies | 1 |
| 3 | " | " | 2 |
| 4 | " | " | 3 |
| 10 | " | " | 10 — 1 |

Thus the number of single *extents* between any number of points in a straight line is one less than the number of points.

We will now calculate the number of single and compound *extents* between a given number of points in a straight line.

2 *points.* Between 2 points lies 1.

3 *points.* From each point to every other point may lie one. Thus from 3 points there would be $3 \times 2 = 6$; but as each line goes out from 2 points, and has therefore been reckoned twice, the true number of *extents* is $\frac{1}{2}(3 \times 2) = 3$.

4 *or more points.* From every point may proceed as many *extents* as there are points beside this one. Therefore *to find the number of single and compound extents in a straight line, multiply the given number of points by the same number less* 1, *and divide this product by* 2.

Between 2 points may lie $\frac{2 \times 1}{2} = 1$

" 3 " " $\frac{3 \times 2}{2} = 3$

" 4 " " $\frac{4 \times 3}{2} = 6$

" 12 " " $\frac{12 \times 11}{2} = 66$

**28.** ***What is the greatest number of diagonals which can be drawn in any polygon?***

*Answer.* 1. A *triangle* can have no diagonal, for the vertex of each angle is already connected with the vertices of each of the others.

2. A *quadrilateral.* One diagonal can be drawn from the vertex of each angle to the vertex of one of the others. But each diagonal connects 2 vertices, therefore there can be only two diagonals in a quadrilateral.

3. In a *pentagon* are 5 vertices; each is connected with 2 others by the bounding lines; therefore from each vertex only 2 diagonals can be drawn; from the 5 vertices $2 \times 5 = 10$; but this is double the real number, which is $\frac{2 \times 5}{2} = 5$.

4. If we extend the examination to *hexagons*, we shall find that so many diagonals can be drawn from the vertex of each angle of a figure, as the figure has sides less 3; each vertex being already connected with 2 others by the bounding lines.

Rule. *To find the number of diagonals that may be drawn in a polygon, multiply the number of its sides or angles by the same number less* 3, *and divide this product by* 2.

In a triangle $\frac{3\,(3-3)}{2} = 0$ diagonals.

quadrilateral $\frac{4\,(4-3)}{2} = 2$ "

pentagon $\frac{5\,(5-3)}{2} = 5$ "

**29.** *Suppose the greatest number of diagonals which can be drawn in a polygon to be* 35, *how many sides has the polygon?*

*Answer.* This number 35 is the half of a product found by multiplying together two numbers, whose difference is 3. We must therefore seek for two numbers whose difference is 3, and whose product is 70. These numbers are 10 and 7. The figure has 10 sides.

**30.** *We will now calculate the whole number of* EXTENTS *which may lie between the points of intersection of straight lines intersecting one another at a given number of points.*

Suppose the lines intersect at the greatest number of points. After finding the greatest number of points (**22**) in which each line is intersected, we must find the number of extents in each line, (**27**,) and multiply this product by the number of lines; we shall thus get the whole number of extents.

*Example.* If 6 straight lines intersect at the greatest number of points, then each line will intersect the other 5 lines. In each line will be 5 points of intersection, and between 5 points lie $\frac{5 \times 4}{2} = 10$ *extents;* thus between the points of intersection of the 6 lines lie $6 \times 10 = 60$ extents.

In 7 straight lines there will be $7 - 1 = 6$ points in each;

$\frac{(7-1)\times(7-2)}{2}$ *extents*; in all 7 lines $\frac{7\times(7-1)\times(7-2)}{2}$ *extents.*

In the same manner,

3 lines give $\frac{3\times(3-1)\times(3-2)}{2}=\frac{3\times2\times1}{2}=3$

4 " " $\frac{4\times(4-1)\times(4-2)}{2}=\frac{4\times3\times2}{2}=12$

5 " " $\frac{5\times(5-1)\times(5-2)}{2}=\frac{5\times4\times3}{2}=30$

6 " " $\frac{6\times(6-1)\times(6-2)}{2}=\frac{6\times5\times4}{2}=60$

10 " " $\frac{10\times(10-1)\times(10-2)}{2}=\frac{10\times9\times8}{2}=360$

20 " " $\frac{20\times(20-1)\times(20-2)}{2}=\frac{20\times19\times18}{2}=3420$

**31.** *We will now compare lines together in regard to their length.*

2 *Lines* are either equal or unequal. Call one A, the other B. Then A is either equal or not equal to B, and in the latter case A may be either longer or shorter than B.

3 *Lines* may all be of equal length.
2 may be equal, 1 not equal.
All may be unequal. 3 cases.

4 *Lines.* All equal.
3 equal—1 not equal.
2 equal—2 not equal, but equal to one another.
2 equal—2 not equal, and not equal to each other.
All unequal. 5 cases.

**6 *Lines.*** All equal.
5 equal—1 unequal.
4 equal—2 unequal, but equal to each other.
4 equal—2 unequal, and unequal to each other.
3 equal—3 unequal, but equal one to another.
3 equal—3 unequal, but 2 equal to each other.
3 equal—3 unequal, and unequal one to another.
2 equal—4 unequal, but equal one to another.
2 equal—4 unequal, but 3 equal one to another.
2 equal—4 unequal, but equal by pairs.
2 equal—4 unequal, and of these 2 are equal to one another, the other 2 unequal.
2 equal—4 unequal, and unequal one to another.
All unequal. 13 cases.

Of these cases some are alike. The 8th is the same with the 3d, and the 9th with the 6th. Thus there are in all 11 different cases.

We can arrive at the same result by numbers without reference to forms. It is only necessary to seek into how many sets 6 can be divided, so that the sum of the numbers of each set shall be 6.

| | |
|---|---|
| $6$ | $3+1+1+1$ |
| $5+1$ | $2+2+2$ |
| $4+2$ | $2+2+1+1$ |
| $4+1+1$ | $2+1+1+1+1$ |
| $3+3$ | $1+1+1+1+1+1$ |
| $3+2+1$ | |

11 Cases.

## III. Angles.

**32.** *We will now seek the number of angles which can be made by a given number of straight lines.*

*Two straight lines* may make 1, 2, or 4 angles, (fig. 16.)

*Three straight lines*, if they meet at 1 point, may make at least 2, at most 6 angles, (fig. 17.)

If they meet at 2 points, they may make at least 2, at most 8 angles; they cannot make 7 angles, because 3 angles cannot be formed at one point by 2 straight lines.

If they meet at 3 points, they make at least 3, at most 12 angles; but never 11 angles.

In these calculations only the simple angles are considered. When three or more lines meet at one point beside the simple angles, others are formed which are the sum of two or more angles, and which may be called compound angles. If three lines meet at one point, beside the two simple angles, a third is formed of these two added together.

**33.** *We will now consider the kind, as well as the number, of the angles which may be made by a given number of straight lines.*

I. *Two straight lines* may make, (fig. 18,)

| | | |
|---|---|---|
| either 1 Right | 2 Right, | 4 Right |
| or 1 obtuse | or 1 ob. and 1 ac. | or 2 ob. and 2 ac. |
| or 1 acute | | |

II. *Three straight lines meeting at* 1 *point.*

1. No line passing beyond the point of meeting.

| | |
|---|---|
| 1 R. and 1 ac. | 3 ob. |
| (fig. 19,) or 1 ob. and 1 ac. | 1 R. 2 ob. (fig. 20.) |
| or 2 ac. | 2 ob. 1 ac. |

2. One line passing beyond the point of meeting, (fig. 21.)

| | |
|---|---|
| 2 ob. 2 ac. | 1 R. 2 ac. |
| 2 R. 1 ob. 1 ac. | 1 ob. 2 ac. |
| 3 ac. | |

3. Two lines passing beyond the point of meeting, (fig. 22.)

1 R. 1 ob. 3 ac.
3 R. 2 ac.
2 ob. 3 ac.
1 ob. 4 ac.

4. All three lines passing beyond the point of meeting, (fig. 23.)

2 R. 4 ac.
2 ob. 4 ac.
6 ac.

III. *Three straight lines meeting at 2 points*, (fig. 24.)

1. No line passing beyond the points of meeting.

| | |
|---|---|
| 2 R. | 2 ob. |
| 1 R. 1 ac. | 1 ob. 1 ac. |
| 1 R. 1 ob. | 2 ac. |

2. When 1 line passes 1 point, (fig. 25.)

| | |
|---|---|
| 3 R. | 1 R. 1 ac. 1 ob. |
| 2 R. 1 ac. | 2 ob. 1 ac. |
| 2 R. 1 ob. | 1 ob. 2 ac. |

3. When 1 line passes 2 points, (fig. 26.)

4 R.
2 R. 1 ob. 1 ac.
2 ob. 2 ac.

4. When 2 lines pass beyond the junction-points, (fig. 27.)

| | | |
|---|---|---|
| 4 R. | 5 R. | 1 R. 2 ob. 2 ac. |
| 2 R. 1 ob. 1 ac. | 4 R. 1 ac. | 3 ob. 2 ac. |
| 2 ob. 2 ac. | 4 R. 1 ob. | 2 ob. 3 ac. |

| | |
|---|---|
| 6 R. | 3 ob. 3 ac. |
| 4 R. 1 ob. 1 ac. | 2 R. 2 ob. 2 ac. |

5. When all the lines pass beyond the junction-points, (fig. 28.)

| | |
|---|---|
| 6 R. | 8 R. |
| 4 R. 1 ob. 1 ac. | 4 R. 2 ob. 2 ac. |
| 2 R. 2 ob. 2 ac. | 4 ob. 4 ac. |
| 3 ob. 3 ac. | |

IV. *Three straight lines meeting at* 3 *points.*

1. When no line passes a junction-point, (fig. 29.)

1 R. 2 ac.
1 ob. 2 ac.
3 ac.

2. When 1 line passes 1 point, (fig. 30.)

| | |
|---|---|
| 2 R. 2 ac. | 2 ac. 2 ob. |
| 1 R. 1 ob. 2 ac. | 1 ob. 3 ac. |

3. When 1 line passes 2 points, (fig. 31.)

| | |
|---|---|
| 2 R. 1 ob. 2 ac. | 3 ob. 2 ac. |
| 1 R. 2 ob. 2 ac. | 2 ob. 3 ac. |

3–2. When 2 lines pass through the same point, (fig. 31–2.)

| | |
|---|---|
| 4 R. 2 ac. | 2 ob. 4 ac. |
| 1 R. 2 ob. 3 ac. | 3 ob. 3 ac. |

4. When 2 lines pass each 1 point, (fig. 32.)

| | |
|---|---|
| 2 R. 1 ob. 2 ac. | 3 ob. 2 ac. |
| 1 R. 2 ob. 2 ac. | 2 ob. 3 ac. |

5. When 1 line passes 2 points and another passes 1, (fig. 33.)

| | |
|---|---|
| 2 R. 2 ob. 2 ac. | 4 R. 1 ob. 2 ac. |
| 3 ob. 3 ac. | 2 R. 2 ob. 3 ac. |
| 1 R. 3 ob. 3 ac. | 3 ob. 4 ac. |
| 4 ob. 3 ac. | |

6. When 2 lines pass 2 points each, (fig. 34.)

4 R. 2 ob. 2 ac.
2 R. 2 ob. 3 ac.
4 ob. 4 ac.

7. When 3 lines pass 1 point each, (fig. 35.)

| | |
|---|---|
| 2 R. 2 ob. 2 ac. | 1 R. 3 ob. 3 ac. |
| 3 ob. 3 ac. | 4 ob. 3 ac. |
| 4 R. 1 ob. 2 ac. | 3 ob. 4 ac. |
| 2 R. 2 ob. 3 ac. | |

8. When 1 line passes 2 points, and 2 lines pass 1 point each, (fig. 36.)

| | |
|---|---|
| 4 R. 2 ob. 2 ac. | 4 R. 2 ob. 3 ac. |
| 2 R. 3 ob. 3 ac. | 1 R. 4 ob. 4 ac. |
| 4 ob. 4 ac. | 5 ob. 4 ac. |
| | 4 ob. 5 ac. |

9. When 2 lines pass 2 points each, and 1 line passes 1 point, (fig. 37.)

4 R. 3 ob. 3 ac.
2 R. 4 ob. 4 ac.
5 ob. 5 ac.

10. When each line passes 2 points, (fig. 38.)

4 R. 4 ob. 4 ac.
6 ob. 6 ac.

**34.** If there be only one angle at a point, we designate it by the letter at the *vertex*, as the point where

the sides meet and form the angle is called. If there be more than one angle at the same point, we make use of three letters, as, (fig. 39,) BAC, CAD, &c., the letter at the vertex of the angle being placed in the middle; or we sometimes use one letter placed within the vertex; as the angle BAC may also be called the angle x.

Now let us suppose the side AC of the acute angle BAC to depart more and more from AB, so as successively to reach AD, AE, and AF. By this movement, the mutual inclination of the sides AB and AC will be changed, and the angle BAC will become greater and greater: thus BAD is a right angle; BAE is an obtuse angle; BAF is in fact no angle at all, though it is sometimes considered as equal to two right angles; for BAD is a right angle; therefore DAF must be a right angle; since by the definition of a right angle they are equal. If we wish to make more than two angles at the point A, on the same side of the line BF, it can only be done by dividing one or both of these right angles, and the whole taken together will only be equal to 2 R. A. In like manner the sum of all the angles which can be made on the other side of the line BF, at the point A, is equal to 2 R. A.; therefore the sum of all the angles, which can be made about the point A, or about any point, is equal to 4 R. A.

Let us suppose the side AC to be moved forward until it coincides with AG; then BAG can be considered as an angle; this angle, which is greater than 2 R. A., is called a *convex* angle. All angles not convex are *concave* angles. Obtuse, right, and acute angles are all concave angles. Wherever there is a concave angle, there will also be a convex angle.

**35.** *We will now seek how many concave and convex*

*angles may be formed by straight lines going out, or* RADIATING *from* 1 *point.*

*Two straight lines* may form 2 angles, viz.: 1 convex and 1 concave.

*Three straight lines* may form 6 angles, viz.: 3 convex and 3 concave.

The whole number of angles formed at a point is double the number of the concave angles. For as there is a convex wherever there is a concave angle, we may reckon two angles at each vertex.

The number of simple and compound angles always equals the number of concave and convex, for the sum of each is the greatest possible number of angles.

*Four straight lines.* Each straight line, or *ray*, may form 2 angles with each of the other rays; therefore each ray $3 \times 2$ angles; 4 rays $4 \times 3 \times 2$ angles. But each angle has 2 sides; therefore the product $4 \times 3 \times 2$ must be divided by 2; thus we have $\frac{4 \times 3 \times 2}{2} = 4 \times 3 = 12$ angles.

We may proceed in a similar manner with any number of rays. We shall find this general rule. *To find the number of angles, both concave and convex, made by a given number of straight lines radiating from one point; Multiply the number of lines by the same number, less* 1.

| | | | |
|---|---|---|---|
| 2 | straight lines | give | $2 \times 1 = 2$ angles. |
| 3 | " | " | $3 \times 2 = 6$ " |
| 5 | " | " | $5 \times 4 = 20$ " |
| 20 | " | " | $20 \times 19 = 380$, &c. |

**36.** *Suppose a given number of simple angles to be formed about a point, let us consider what kind of angles they may be.*

1. *Of two angles*, one will be a concave angle less than 2 R. A.; the other a convex angle greater than 2 R. A. As much as the one is less than 2 R. A., by just so much will the other be greater than 2 R. A., since the sum of both is equal to 4 R. A. For the same reason there can only be one convex angle about a point.

2. *Three angles.* There may be 3 concave, or 2 concave and 1 convex. If the 3 concave angles are equal, each is an obtuse angle; if they are not equal, there may be either 3 obtuse, or 2 obtuse and 1 R. A., or 2 ob. and 1 ac.

3. *Four angles.* There may be 4 concave, or 3 concave and 1 convex. The 4 concave angles may be as follows:

| | |
|---|---|
| 4 R. A. | 3 ob. 1 ac. |
| 2 R. A. 1 ob. 1 ac. | 2 ob. 2 ac. |
| 1 R. A. 2 ob. 1 ac. | 1 ob. 3 ac. |
| 1 R. A. 1 ob. 2 ac. | 7 cases. |

**37.** ***We will now consider the kind of concave angles, which different polygons may have.***

1. *The Quadrilateral.*

| | |
|---|---|
| 4 R. A. | 3 ob. 1 ac. |
| 2 R. A. 1 ob. 1 ac. | 2 ob. 2 ac. |
| 1 R. A. 2 ob. 1 ac. | 1 ob. 3 ac. |
| 1 R. A. 1 ob. 2 ac. | 7 cases, (fig. 40.) |

2. *The Pentagon.*

| | |
|---|---|
| 5 ob. | 3 ob. 2 ac. |
| 4 ob. 1 R. A. | 2 ob. 3 R. A. |
| 4 ob. 1 ac. | 2 ob. 2 R. A. 1 ac. |
| 3 ob. 2 R. A. | 2 ob. 1 R. A. 2 ac. |
| 3 ob. 1 R. A. 1 ac. | 2 ob. 3 ac. 10 cases. |

3. *The Hexagon.* Of this there may be 10 different cases.

6 ob. 1R. 5ob 5ob. 1ac
2R. 4ob 1R. 4ob 1ac 4ob. 2ac
2R. 3ob 1ac. 1R. 3ob 2ac. 3ob. 3ac
3R. 3ob.

**38.** *What is the greatest number of convex angles which a rectilineal figure may have?*

A convex angle, when it occurs in a figure, is called a *re-entering* angle, because the vertex is directed inwards; the concave angles are called *salient* angles. In all the rules and remarks, figures with *salient* angles only are intended, except the contrary is expressly stated.

A *triangle* can have no convex angle.

A *quadrilateral* can have 1 convex angle.

A *pentagon* " " 2 " "

A *hexagon* " " 3 " "

In general, in any rectilineal figure there may be, at the most, as many *convex* (or re-entering) angles as the figure has sides less 3.

In every rectilineal figure there must be at least 3 concave angles.

## IV. Figures.

**39.** *What is the greatest number of triangles which can be formed with a given number of straight lines?*

1. *Three straight lines* can form 1 triangle, neither more nor less.

2. *Four straight lines* may intersect one another at 6 points, (**22.**) In each line there would be 3 intersection-points, and between 3 points there may be 3 extents, (**27.**) Upon each *extent* there may be one triangle, and thus there would be as many triangles as there are *extents.* The number of extents is $3 \times 4 = 12$. But each triangle has 3 sides, and therefore we have reckoned each triangle 3 times in the above estimate. We must therefore

divide the product 12 by 3 to obtain the real number of triangles. It is $\frac{12}{3}=4$.

3. *5 straight lines.* If 5 straight lines intersect one another at the greatest number of points, there will be in each straight line 4 intersection-points. Between 4 points there may be $\frac{4\times 3}{2}$ *extents;* therefore, in the 5 lines there are $\frac{5\times 4\times 3}{2}$ extents. Upon each of these *extents* may be one triangle, and thus $\frac{5\times 4\times 3}{2}$ triangles. But in this calculation each triangle is counted 3 times, therefore the true number will be $\frac{5\times 4\times 3}{2\times 3}$ or $\frac{5\times 4\times 3}{1\times 2\times 3}$ $=10$.

From these examples we may deduce the rule. *To find the greatest number of triangles which may be formed by a given number of straight lines, multiply the number of the lines by the product of the two numbers preceding it in the natural order of the numbers, and divide the product thus obtained by* $1\times 2\times 3$.

7 straight lines can form at most $\frac{7\times 6\times 5}{1\times 2\times 3}=35$ triangles.

8 " " " $\frac{8\times 7\times 6}{1\times 2\times 3}=56$ "

20 " " " $\frac{20\times 19\times 18}{1\times 2\times 3}=1140$ "

**40.** *What number of parallelograms can be formed by two sets of parallel lines crossing each other?* (figs. 41 and 42.)

1. 3 *parallels in one set, and* 2 *in the other set.* There will be as many parallelograms as there are inter-

vals between the 3 parallels; for all the intervals will be enclosed by the 2 intersecting parallels. Between the 3 parallels there will be as many intervals, single and compound, as there are extents between 3 points in a straight line, that is $\frac{3\times 2}{2}=3$. Consequently, in the supposed case, 3 parallelograms are formed.

2. 4, 5, 6, *or more parallels in one set, and* 2 *parallels in the other set.* Find as before the number of intervals, single and compound, in each set, and multiply them together. But 2 parallels give $\frac{2\times 1}{2}=1$ interval; we have therefore only to find the number of intervals in the other set, and it will be the required number of parallelograms.

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 parallels | crossed | by | 2 parallels | give | $\frac{4\times 3}{2}=6$ | parallel'ms. |
| 5 | " | " | 2 | " | $\frac{5\times 4}{2}=10$ | " |
| 6 | " | " | 2 | " | $\frac{6\times 5}{2}=15$ | " |

From these examples, the rule for finding the number of parallelograms formed by the intersection of any number of parallel lines by 2 *parallel lines* is apparent. *Multiply the number of the lines of the former set by the same number less* 1, *and divide this product by* 2.

3. Let there be any number of parallels in each set, and we have this rule for finding the number of parallelograms made by the intersection of the two sets. *Multiply the number of intervals between the parallels of one set, by the number of intervals between the parallels of the other.*

10 parallels crossed by 10 parallels give

$$\frac{10\times 9}{2}\times\frac{10\times 9}{2}=2025 \text{ parallelograms.}$$

6 parallels crossed by 4 parallels give

$$\frac{6 \times 5}{2} \times \frac{4 \times 3}{2} = 90 \text{ parallelograms.}$$

12 parallels crossed by 30 parallels give

$$\frac{12 \times 11}{2} \times \frac{30 \times 29}{2} = 28.710 \text{ parallelograms.}$$

The rule may be thus expressed: *Multiply the number of lines in each set taken separately by the same number less* 1; *take half of each product; multiply these half-products together, and this last product will be the number required.*

**41.** We will now set down in a tabular form the number of parallelograms formed by intersecting sets of parallel lines. It may be done in the familiar manner of the multiplication table. Set down the numbers for one set in a horizontal series, and the numbers for the other set in a vertical series, and the number of parallelograms formed by the intersection of the 2 sets directly under the one, and opposite to the other.

| Parallels | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 1 | 3 | 6 | 10 | 15 | 21 |
| 3 | 0 | 3 | 9 | 18 | 30 | 45 | 63 |
| 4 | 0 | 6 | 18 | 36 | 60 | 90 | 126 |
| 5 | 0 | 10 | 30 | 60 | 100 | 150 | 210 |
| 6 | 0 | 15 | 45 | 90 | 150 | 225 | 315 |
| 7 | 0 | 21 | 63 | 126 | 210 | 315 | 441 |

The numbers of the first vertical series are the same as those of the first horizontal series; the numbers of the second vertical series are the same as those of the second horizontal series, and so on.

The differences between each two successive numbers of the horizontal and of the vertical series are as follows:

| | | | | | | |
|---|---|---|---|---|---|---|
| The First | 0 | 0 | 0 | 0 | 0 | 0 |
| Second | 1 | 2 | 3 | 4 | 5 | 6 |
| Third | 3 | 6 | 9 | 12 | 15 | 18 |
| Fourth | 6 | 12 | 18 | 24 | 30 | 36 |
| Fifth | 10 | 20 | 30 | 40 | 50 | 60 |
| Sixth | 15 | 30 | 45 | 60 | 75 | 90 |
| Seventh | 21 | 42 | 63 | 84 | 105 | 126 |

Again, remark the difference between each two successive numbers of the horizontal and also of the vertical series in this table.

The horizontal series give these differences, viz.:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1st | horizontal | series | 0 | 0 | 0 | 0 | 0 | 0 |
| 2d | " | " | 1 | 1 | 1 | 1 | 1 | 1 |
| 3d | " | " | 3 | 3 | 3 | 3 | 3 | 3 |
| 4th | " | " | 6 | 6 | 6 | 6 | 6 | 6 |
| 5th | " | " | 10 | 10 | 10 | 10 | 10 | 10 |
| 6th | " | " | 15 | 15 | 15 | 15 | 15 | 15 |

The vertical series give these differences, viz.:

| 1st. | 2d. | 3d. | 4th. | 5th. | 6th. |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 4 | 6 | 8 | 10 | 12 |
| 3 | 6 | 9 | 12 | 15 | 18 |
| 4 | 8 | 12 | 16 | 20 | 24 |
| 5 | 10 | 15 | 20 | 25 | 30 |
| 6 | 12 | 18 | 24 | 30 | 36 |

Once more, remark the difference between the successive numbers of the series last formed.

In the horizontal they are as follows, viz.:

| | | | | |
|---|---|---|---|---|
| 1st. | 1 | 1 | 1 | 1 |
| 2d. | 2 | 2 | 2 | 2 |
| 3d. | 3 | 3 | 3 | 3 |
| 4th. | 4 | 4 | 4 | 4 |
| 5th. | 5 | 5 | 5 | 5 |
| 6th. | 6 | 6 | 6 | 6 |

In the vertical they are

| 1st. | 2d. | 3d. | 4th. | 5th. | 6th. |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 2 | 3 | 4 | 5 | 6 |

*Note.* The preceding tables of differences are given as an example of the manner in which a subject should be treated. The scholar should be accustomed to examine a subject in all its relations.

**42.** *Into how many triangles can a rectilineal figure be divided by diagonals not intersecting one another?*

The vertex of each angle is already connected by the bounding lines with the vertices of 2 other angles; there remain then just as many vertices with which each vertex is not connected, as the figure has sides less 3. Consequently there may be as many diagonals not intersecting one another, as the figure has sides less 3.

1. *The triangle.* This can have no diagonal; for $3-3=0$.

2. *The quadrilateral.* Here we have $4-3=1$ diagonal, which will divide the figure into 2 triangles.

3. *The pentagon.* Here we have $5-3=2$ diagonals, dividing the figure into 3 triangles.

In each case we have one more triangle than we have diagonals, and hence we may infer the general rule. *If as many diagonals be drawn in a rectilineal figure as can be without one intersecting another, there will be formed as many triangles as the figure has sides less 2.*

| | | |
|---|---|---|
| In a quadrilateral | $4-2=2$ | triangles. |
| In a hexagon | $6-2=4$ | " |
| In a decagon | $10-2=8$ | " &c. |

**43.** *We will now consider the number and kind of figures into which a rectilineal figure may be divided by straight lines drawn at pleasure.*

1. *The triangle*, (fig. 43.) A line drawn from the vertex of an angle to the opposite side divides the triangle into 2 triangles. A line drawn from one side to another side divides the triangle into 1 triangle and 1 quadrilateral.

2. *The quadrilateral*, (fig. 44.) A line drawn from the vertex of an angle may meet a side, or the vertex of another angle. In the former case 1 triangle and 1 quadrilateral, in the latter case 2 triangles, are formed.

A line drawn from one side of the quadrilateral may meet either the adjacent or the opposite side. In the former case 1 triangle and 1 pentagon, in the latter case, 2 quadrilaterals, are formed. Fig. 45.

*Note.* In the preceding exercises one or two examples only have been given, to show the manner of proceeding. Each teacher can increase the number at his pleasure.

## V. Solids.

**44.** *We will now find by calculation the number of lines and angles in a prism.*

I. *Lines.* 1. Edges or Sides.

The triangular prism has $\frac{2\times3+3\times4}{2}=\frac{6\times3}{2}=3\times3$

quadrangular " $\frac{2\times4+4\times4}{2}=\frac{6\times4}{2}=3\times4$

pentagonal " $\frac{2\times5+5\times4}{2}=\frac{6\times5}{2}=3\times5$

hexagonal " $\frac{2\times6+6\times4}{2}=\frac{6\times6}{2}=3\times6$

Rule. *Multiply the number of sides in the base by* 3.

2. Axes.

| | Corner Axes. | Edge Axes. | Face Axes. | Corner Edge Axes. | Edge Face Axes. |
|---|---|---|---|---|---|
| The triangular prism | 0 | 0 | 1 | 2×3 | 3 |
| quadrangular " | 4 | 6 | 3 | 0 | 0 |
| pentagonal " | 0 | 0 | 1 | 2×5 | 5 |
| hexagonal " | 6 | 9 | 4 | 0 | 0 |
| heptagonal " | 0 | 0 | 1 | 2×7 | 7 |
| octagonal " | 8 | 12 | 5 | 0 | 0 |
| nonagonal " | 0 | 0 | 1 | 2×9 | 9 |
| decagonal " | 10 | 15 | 6 | 0 | 0 |

II. *Angles.* 1. Line Angles.

Triangular prism has $6\times3=18$
quadrangular " " $8\times3=24$
pentagonal " " $10\times3=30$
decagonal " " $20\times3=60$

2. Plane Angles.

Triangular prism has $\frac{6\times3}{2}=9$

quadrangular " " $\frac{8\times3}{2}=12$

pentagonal " " $\frac{10\times3}{2}=15$

decagonal " " $\frac{20\times3}{2}=30$

3. Solid Angles.

| | | |
|---|---|---|
| Triangular prism has | | $2 \times 3 = 6$ |
| quadrangular " | " | $2 \times 4 = 8$ |
| pentagonal " | " | $2 \times 5 = 10$ |
| decagonal " | " | $2 \times 10 = 20$ |

That is, a prism has twice as many solid angles as its base has sides.

**45.** *We will now find the number of lines and angles in a pyramid.*

I. *The Lines.* Edges or Sides.

Triangular pyramid has $\frac{(3 \times 3) + 3}{2} = 6$

quadrangular " " $\frac{(4 \times 3) + 4}{2} = 8$

pentagonal " " $\frac{(5 \times 3) + 5}{2} = 10$

decagonal " $\frac{(10 \times 3) + 10}{2} = 20$

II. *Angles.* 1. Line Angles.

| | | |
|---|---|---|
| Triangular pyramid has | | $(3 \times 3) + 3 = 12$ |
| quadrangular " | " | $(4 \times 3) + 4 = 16$ |
| pentagonal " | " | $(5 \times 3) + 5 = 20$ |
| decagonal " | " | $(10 \times 3) + 10 = 40$ |

2. Plane Angles.

Triangular pyramid has $\frac{(3 \times 3) + 3}{2} = 6$

quadrangular " " $\frac{(4 \times 3) + 4}{2} = 8$

pentagonal " " $\frac{(5 \times 3) + 5}{2} = 10$

decagonal " " $\frac{(10 \times 3) + 10}{2} = 20$

3. Solid Angles.

| | |
|---|---|
| Triangular pyramid has | $3+1=4$ |
| quadrangular " " | $4+1=5$ |
| pentagonal " " | $5+1=6$ |
| decagonal " " | $10+1=11$ |

**46.** *We will now find by calculation the number of lines and angles in the regular solids.*

I. *Lines.* 1. Exterior. Edges or Sides.

| | |
|---|---|
| Tetraedron has | $\frac{4\times 3}{2}=6$ |
| Octaedron " | $\frac{8\times 3}{2}=12$ |
| Icosaedron " | $\frac{20\times 3}{2}=30$ |
| Hexaedron " | $\frac{6\times 4}{2}=12$ |
| Dodecaedron " | $\frac{12\times 5}{2}=30$ |

2. Interior. Axes.

| | Corner Axes. | Face Axes. | Edge Axes. | Corner Face Axes. |
|---|---|---|---|---|
| Tetraedron has | 0 | 0 | $\frac{6}{2}=3$ | 4 |
| Octaedron " | $\frac{6}{2}=3$ | $\frac{8}{2}=4$ | $\frac{12}{2}=6$ | 0 |
| Icosaedron " | $\frac{12}{2}=6$ | $\frac{20}{2}=10$ | $\frac{30}{2}=15$ | 0 |
| Hexaedron " | $\frac{8}{2}=4$ | $\frac{6}{2}=3$ | $\frac{12}{2}=6$ | 0 |
| Dodecaedron " | $\frac{20}{2}=10$ | $\frac{12}{2}=6$ | $\frac{30}{2}=15$ | 0 |

II. *Angles.*

| | Line Angles. | Face Angles. | Solid Angles. |
|---|---|---|---|
| Tetraedron has | $4 \times 3 = 12$ | $\frac{4 \times 3}{2} = 6$ | $\frac{4 \times 3}{3} = 4$ |
| Octaedron " | $8 \times 3 = 24$ | $\frac{8 \times 3}{2} = 12$ | $\frac{8 \times 3}{4} = 6$ |
| Icosaedron " | $20 \times 3 = 60$ | $\frac{20 \times 3}{2} = 30$ | $\frac{20 \times 3}{5} = 12$ |
| Hexaedron " | $6 \times 4 = 24$ | $\frac{6 \times 4}{2} = 12$ | $\frac{6 \times 4}{3} = 8$ |
| Dodecaedron " | $12 \times 5 = 60$ | $\frac{12 \times 5}{2} = 30$ | $\frac{12 \times 5}{3} = 20$ |

# PART SECOND.

## SECTION SECOND.

## CONSTRUCTION.

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### I. Lines.

**47.** Hitherto we have drawn figures on the board, and on the slate, by the eye, without regard to strict accuracy; we will now make use of instruments, and draw them with more care.

To draw straight lines we make use of a ruler. That we may be accustomed to draw them in all positions of the ruler, we will take points in various parts of the paper, (or slate,) and connect them by straight lines, keeping the paper (or slate) always in the same position.

**48.** To draw circular lines we make use of a pair of compasses. These must be opened to a certain distance, the point of one leg fixed tight in the paper, and the point of the other leg moved on the surface of the paper about the fixed point, until the curved line thus made shall return into itself, and form a complete figure. This curved line is the circumference of a circle; the point of the fixed leg is the centre of the circle. The size of the circle depends upon the greater or less distance between the points of the compasses. The radius

of the circle is exactly equal to this distance: hence, to express the size of a circle, we give the length of its radius; thus we say a circle of 1 inch, 1 foot, or 1000 feet radius.

Many circles may be *described*, that is drawn, about the same centre; and their circumferences will remain in all parts at the same distance one from another. Two such circles, besides having a common centre, have the surface of the smaller in common, and the difference between their surfaces is an *annular* surface or ring.

**49.** Two circles in a plane may have various relative positions. They may have a common centre, or they may not. In the latter case the circles may lie entirely apart, or their circumferences may meet at 1 point on the outside, or may intersect at 2 points, so that they shall have in common a surface enclosed by 2 arcs. One circle may be entirely within the other, without having a common centre; of this there may be two cases, viz., the circumferences may be entirely separate, or they may touch at 1 point.

We will draw several figures and examine them. (Fig. 46.)

In 1 the distance between the centres of the circles = 0.

In 2 the distance is greater than the *sum* of the 2 radii, by so much of the straight line joining the 2 centres as lies between the 2 circumferences.

In 3 the distance between the 2 centres is equal to the sum of the 2 radii.

In 4 the distance is less than the sum of the 2 radii by so much of the straight line joining the 2 centres as lies between the intersecting arcs.

In 5 the distance is less than the difference of the 2

radii by so much of the radius of the greater circle passing through the centre of the lesser, as is contained between the 2 circumferences.

In 6 the distance is exactly equal to the difference of the radii.

**50.** *Suppose it is required to draw* 10 *straight lines, of which the* 1*st shall be* 1 *inch long, the* 2*d* 2 *inches long, and the* 3*d* 3 *inches long, and so on, the* 10*th being* 10 *inches long.*

Draw a straight line of any length; open the compasses so that the points shall be 1 inch apart, or, as it is more concisely expressed, take 1 inch between the points of the compasses, and apply them to the line. To get the 2d line, apply the compasses twice continuously to the line already drawn; or take 2 inches between the points and apply them once. Proceed in this manner; for the 10th line apply an opening of 1 inch 10 times continuously.

If it is required to draw a straight line which shall be 2, 3, 4 or more times as long as another line; then draw a line of any length; take this length between the points of the compasses, and apply it as often as is required to a line of indefinite length.

How shall we cut from a long line a part equal to a shorter one? Take the length of the shorter one between the points of the compasses; apply them to the longer line, placing one point at the end of the line. The remainder of the long line is the *difference* between the lines.

**51.** *Suppose it is required to describe from the ends of the line* AB, *as centres,* 2 *circles which shall have the following relations one to the other.*

1. *Having nothing in common*, (fig. 46. 2.) Fix one leg of the compasses at A, and with an opening less than AB, for example = AC, describe a circle. Then fix one leg of the compasses at B, and with an opening less than BC describe a circle. We have the 2 circles required.

2. *The surfaces having no part in common, but the circumferences touching at* 1 *point*, (fig. 46. 3.) From A, as a centre, with an opening of the compasses, or, (to speak more technically,) a radius, less than AB, for example = AC, describe a circle; from B, as a centre, with a radius = BC, describe another circle; we have the 2 circles required, which will touch each other externally.

3. *The circumferences touching at* 2 *points*, (fig. 46. 4.) From A, as a centre, with a radius less than AB, describe a circle; from B, as a centre, with a radius greater than BC, but less than B~~A~~, describe another circle; we have the circles required. E Suppose the line BA to be continued to E, it is evident that the radius of the circle described from B may be taken either equal to or greater than BA. If it be taken = BE = BA + AE = BA + AC, (since AE and AC are equal, being radii of the same circle,) then the first circle will meet the 2d at one point only, viz. at E.

4. *Having the surface of one in common, without the circumferences touching each other*, (fig. 46. 5.) Produce AB to C. From A, as a centre, with a radius AC, describe a circle; from B, with a radius less than BC, for example = BD, describe another circle. We have the circles required.

5. *Having the surface of one in common, and the circumferences touching at* 1 *point*, (fig. 46. 6.) Produce AB to any point C. From A, with a radius = AC,

describe a circle, and from B, with a radius $=$ BC, describe another circle.

**52.** *To divide a straight line* AB *into* 2, 4, 8, 16, *&c., equal parts*, (fig. 47.)

From A and B, as centres, with equal radii, which must be greater than ½ AB, describe two circles; the circumferences of these circles will intersect at 2 points, C and D, one on each side of AB. Join C and D by a straight line. The line AB will be bisected at the point where CD crosses it. It is not necessary to describe entire circumferences; 2 intersecting arcs on each side of AB will be sufficient. If the line is very long, or the compasses very small, we can take any 2 points in the line for centres, so they are at equal distances from the ends of the line.

By a similar process bisect each half of AB, and the whole line will be divided into 4 equal parts, (fig. 48.) Bisect each of these 4 parts, and the whole line will be divided into 8 equal parts.

**53.** *To draw a curved line like a steel spring*, (fig. 49.)

Draw a straight line, and take in it any point A. From A, as a centre, with a small radius AB, describe a semi-circumference BC; then from B, as a centre, with radius BC, describe a second semi-circumference below the line AB; then from A, with radius AD, describe a third semi-circumference above the line AB; again from B, with a radius BE, describe a fourth semi-circumference, EF, below the line, and so on. There will thus be formed from connected semi-circumferences a connected curved line. From the construction it is evident that AB $=$ AC, BC $=$ BD, and

AD = AE, consequently BD = CE; BE = BF, consequently CE = DF, thus BC = BD = CE = DF.

**54.** *To draw a curved line winding like a snail-shell,* (fig. 50.)

Draw a straight line and take in it a point A. On each side of this point measure off in the line several small equal parts, for example 3, viz., AD, DH, HB, AC, CG, GL. From A, as a centre, with a radius AB, describe a circumference. Then, on each side of the line AB alternately, describe semi-circumferences, viz., from C, with radius CB; from D, with radius DE; from G, with radius GF. The required curved line will thus be formed. From the diagram it appears that LE contains 2 of the above equal parts; BF has 4; EI has 6.

**55.** *To draw a serpentine line,* (fig. 51.)

Divide a straight line into any number of equal parts, for example, 12. From the 1st, 3d, 5th, 7th, 9th, and 11th points of division, as centres, with a radius equal to one part, describe semi-circumferences, alternately above and below the straight line. A curved line drawn in this manner resembles that made by a snake in motion, and is therefore called a *serpentine* line.

**56.** *To draw a line which curves like the waves,* (fig. 52.)

Draw 3 parallel straight lines at equal distances from one another. We can do this with a ruler, for if it is well made the two edges will be parallel; draw lines along both edges, and we shall have 2 of the parallels; move the ruler and place one edge exactly upon one of the lines already drawn, then draw a line along the

other edge, and we shall have 3 parallels. Divide these parallels into any number of equal parts, for example, 10. Then from the points of division 1, 3, 5, 7, 9, alternately on one and the other of the exterior parallels, with a radius equal to the distance between the first division point and the end of the middle parallel, describe arcs. These will be alternately above and below the middle parallel, and will form a continuous curved line, called a *waving* line.

**57.** *To draw an Ellipse,* (fig. 53.)

Divide a straight line AD into 3 equal parts at the points B and C. From B, with radius BC, describe a circle; and from C, with radius CB, describe another circle; the circumferences of these circles will intersect at 2 points, E and F. From E draw the diameters EG and EH; and from F draw the diameters FI and FK. From E, as a centre, with radius EG, describe the arc GH, and from F, with radius FI, describe the arc IK. Thus from the 4 exterior arcs IK, KH, HG, and GI, is formed a connected figure called an *ellipse.*

**58.** *To draw an Oval,* (fig. 54.)

Describe a circumference and divide it into 4 equal parts; join the opposite division points by the diameters BC and DE; draw CD and BD, and produce them beyond D. From C, with radius CB, describe the arc BF; and from B, with radius BC, describe the arc CG; and from D, with radius DF, describe the arc FG. The arcs BC, CG, GF, and FB form, together, a figure shaped like an egg.

## II. Angles.

**59.** Before proceeding to the construction of angles, it will be well to make ourselves acquainted with the relation between angles and arcs. We have before seen that as the side AC (fig. 39) departs from AB, the angle which it makes with AB is constantly increasing. Now let us suppose that each point in AC describes at the same time an arc of a circle. It is evident that such arcs bear a certain fixed relation to the angle and to one another, since all are made by one and the same motion of the side AC, and begin, increase, and end simultaneously. These arcs differ in actual length; but each is the same fractional part of a whole circumference; if we suppose each circumference to be divided into the same number of equal parts, these arcs will contain an equal number of such parts.

It is usual to divide a circumference into 360 equal parts called *degrees*, and marked thus (°); for example, 40° is read forty degrees. As all circumferences, whether of great or of small circles, are divided into 360°, it follows that a degree is not a fixed quantity, but varies for every different circumference. It merely expresses the magnitude of an arc as compared with the whole circumference of which it is a part, and not with any other circumference. Each degree is divided into 60 equal parts called *minutes*, and marked (′). Each minute is divided into 60 equal parts called *seconds*, marked (″). The division is sometimes carried to *thirds* and *fourths*, marked (‴) (⁗).

We are thus furnished with a very convenient method of measuring angles. As the magnitude of an angle has

no reference to the length of its sides, but to their mutual inclination, or the opening between them, either of the arcs described from its vertex as a centre, and *intercepted*, (taken between,) by its sides, may be taken as the measure of the angle, for they all contain the same number of degrees; which number of degrees denotes the size of the angle.

**60.** If the side AC be moved entirely round the point A, it will have made 4 R. A.; and at the same time each point in it will have described an entire circumference; thus a circumference, or 360°, is the measure of 4 right angles; and therefore a quarter of a circumference, or *quadrant*, as it is called, is the measure of 1 right angle; that is, the sides of every right angle will intercept an arc of $\frac{360°}{4} = 90°$ in the circumference of a circle described from its vertex as a centre with any radius. We say therefore that a right angle is an angle of 90°; half a right angle is an angle of 45°; one third of a right angle is an angle of 30°; two thirds of a right angle is an angle of 60°, &c.

**61.** Having made ourselves acquainted with the principle upon which the mensuration of angles depends, we will now examine the *protractor*, (fig. 55,) which is an instrument used in *plotting*, that is, drawing upon paper angles whose magnitude is known; or for measuring angles already drawn upon paper. It is a semicircle of wood, metal, or horn, accurately divided into 180°. For the convenience of reckoning both ways, the degrees are numbered from the left towards the right, and from the right towards the left. The division lines are all drawn from a point in the middle of the diameter,

called *the centre of the protractor.* This point is marked by a notch in the diameter.

**62.** Another instrument, called a *square*, (fig. 56,) is also used for drawing and measuring right angles. This consists of two rulers fixed together at right angles with one another. Sometimes the ends of these two are connected by a third ruler, thus forming a right-angled triangle. We may ascertain if a *square* is accurately made in this very simple manner, (fig. 57.) Draw a straight line AB; divide it into 2 parts at the point O. Apply one edge of the square to the part AO, placing the vertex of the right angle at O; then, by drawing a line along the other edge, make the angle AOC. Reverse the square, keeping the vertex of the right angle at O, but applying one edge to the other part of the line AB; viz. to OB; draw a line along the other edge, and the angle BOD will be made. If OC and OD do not coincide, the sides of the square do not form exactly a right angle. The angle COD will be twice the difference between the angle made by the 2 sides and a right angle.

**63.** Having examined the construction of these instruments, we will now proceed to use them.

*To make a right angle.*

Solution 1. *With the Protractor.* It is required to make a right angle at the point O in the straight line AB, (fig. 48.) Apply the diameter of the protractor to the given line so that the centre shall fall exactly on the point O. Mark on the paper the point where the division 90° on the arc of the protractor falls. Suppose at C; draw the straight line CO. This line will be perpendicular to AB, and will make 2 R. A. with it.

Sol. 2. ***With the Square.*** Draw straight lines along the 2 sides which make the right angle, and at their point of junction a right angle will be made upon the paper. If one of the sides of the required angle be given, apply one side of the square to this given line, and draw a line along the other side.

Sol. 3. ***With the Compasses***, (fig. 48.) Draw a straight line; take in it any point O, and make OA = OB. From A and B as centres with equal radii describe 2 arcs which will intersect at C. Draw CO. The angles COA and COB are right angles.

In the solutions of this problem we have solved another problem also; namely, that of erecting a perpendicular to a given straight line; for the line CO is perpendicular to AB.

**64.** *To draw a perpendicular from a given point* C *to a given straight line* AB, (fig. 48.)

From the point C, with a radius greater than the shortest distance from C to the line AB, describe an arc which will cut AB at two points, [illegible] and L. From each of these points, with the same radius, describe an arc. These arcs will intersect each other at some point; suppose at M. A line drawn from C to AB, passing through M, will be the perpendicular required.

**65.** *To make an angle the size of which is given in degrees.*

*Sol.* Draw a straight line. Apply to it the diameter of the protractor. Mark on the paper the point where the centre of the protractor lies, and also the point where the given number of degrees on the arc of the protractor lies. Connect these two points by a straight line, and the required angle will be made.

**66.** *To make an angle equal to a given angle* AOB, (fig. 58.)

Sol. 1. *With the Protractor.* Measure the number of degrees in the given angle, and then make an angle of the same number of degrees.

Sol. 2. *With the Compasses.* Draw the straight line CD. From O as a centre, with any radius, describe an arc which shall intersect the sides of the angle AOB. From C as a centre, with an equal radius, describe an arc, which will intersect the line CD at D. From D as a centre, with a radius equal to the chord AB, describe an arc which will intersect the former at the point E. Draw EC. The angle ECD = AOB.

**67.** *Through a given point* C, *to draw a line parallel to a given line* AB, (fig. 61.)

*Sol.* 1. From the point C let fall upon AB the perpendicular DC. Upon DC at C erect the perpendicular CF, which is the line required.

*Sol.* 2. Draw from C a straight line meeting AB at any point E. At C make an angle ECF = ang. CEB. CF is the line required.

**68.** *To divide a given angle* O *into* 2, 4, 8, 16, *&c. equal angles*, (fig. 59.)

*Sol.* From O as a centre, with any radius, describe an arc AB. From A and B as centres with equal radii describe 2 arcs which will intersect at C. Draw CO, and the angle AOB will be divided into 2 equal angles, COA and COB.

Repeat this operation with each of the angles COA and COB, and the entire angle at O will be divided into 4 equal angles. By continuing the process it may be divided into 8, 16, 32, &c. equal angles.

**69.** *To make an angle which shall be* 2, 3, 4, *times, or* $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$ *as great as the angle* O, (figs. 59 and 60.)

*Sol.* From the point M, at which the vertex of the new angle is to lie, draw a straight line of any length. From M as a centre, with radius equal to OB, describe an arc, which will intersect the straight line at N. Beginning at N measure on this arc 2, 3, or 4 times the length of AB; for example, NP, PS, &c.; draw SM, and we shall have the angle SMN = 2 AOB. Or, divide the arc AB into 2, 3, or more equal parts, and measure one or more of these parts on the arc NS; for example, NR = $\frac{2}{3}$ AB; draw RM; the angle RMN = $\frac{2}{3}$ AOB.

## III. Figures.

### CONSTRUCTION OF TRIANGLES.

X **70.** *To construct a triangle which shall be equal to another triangle* ABC, (fig. 62.)

1. *Let the* 3 *sides of* ABC *be given.* Draw a straight line EF equal to one of the given sides, for example, BC. From E as a centre, with radius equal to a second side AB, describe an arc; and from F, with a radius equal to the third side CA, describe another arc which will intersect the former at D. Draw the straight lines DE and DF. DEF is the required triangle.

2. *Suppose the side* BC *and the two adjacent angles* B *and* C *to be known.* Draw a straight line EF = side BC. At the point E make the angle DEF = CBA, at point F make the angle DFE = ACB. Produce the sides of the angles E and F until they intersect at D. DEF is the required triangle.

3. *Suppose the sides* BC *and* BA, *and the included angle* ABC, *to be known.* Draw a straight line EF = side BC. At point E make an angle DEF = CBA. Take ED = BA, and draw DF. DEF is the triangle required.

That the triangles thus constructed are in each case equal to the given triangle will be demonstrated hereafter (**126, 127, 137.**) By *equal* figures we mean figures which coincide entirely when one is laid upon the other; therefore equal figures must be similar.

**71.** *To construct triangles in and about circles.*

1. *An equilateral triangle,* (fig. 63.) Divide the circumference of a circle with the compasses into 3 equal parts. Join the division points by straight lines, and we shall thus construct an equilateral triangle in a circle.

In an equilateral triangle constructed about a circle the sides are tangents to the circle at the above mentioned division points of the circumference. To obtain these tangents draw radii to the division points; and upon the ends of these radii erect perpendiculars. Produce these perpendiculars both ways until they intersect one another, and we shall have an equilateral triangle constructed about a circle.

2. *A scalene triangle.* Divide the circumference of a circle into three unequal parts. Join the division points by straight lines, and we shall thus have constructed a scalene triangle in a circle. Draw tangents to the circle at the division points, and produce such tangents in both directions until they intersect one another, and we have a scalene triangle about a circle.

*Remark.* The vertices of all the angles of each triangle constructed in a circle lie in the circumference of the circle. A figure, the vertices of whose angles are thus situated, is said to be *inscribed in a circle;* and the circle is said to be *circumscribed about the figure.* A figure, all the sides of which are tangents to a circle, is said to be *circumscribed about a circle,* and the circle to be *inscribed in the figure.*

**72.** *To construct an equilateral triangle upon the line* AB, (Fig. 64.)

*Sol.* From A as a centre, with a radius equal to AB, describe a circle; and from B, with a radius BA, describe another circle. The circumferences of these circles will intersect at C and D. Draw the straight lines CA, CB, DA, DB. Two equilateral triangles will thus be constructed, which will be equal to each other.

**73.** *To construct an isosceles triangle upon the line* AB, (fig. 65.)

*Sol.* From A and B as centres, with equal radii, describe two arcs intersecting at C. Draw the straight lines CA and CB. An isosceles triangle will thus be formed, of which AB is the base.

**74.** *To construct upon the line* AB *an isosceles triangle, whose equal sides shall be equal to a given line* M, (fig. 65.)

*Sol.* From A and B as centres, with a radius equal to M, describe two arcs intersecting at C. Draw CA and CB. CAB is the triangle required.

**75.** *To construct a right-angled isosceles triangle upon the line* AB, *with a right angle adjacent to the side* AB, (fig. 66.)

*Sol.* At either of the points A or B erect a perpendicular to the line AB. In this perpendicular take BC = BA. Draw CA, and the required triangle will be constructed.

**76.** *To construct a right-angled isosceles triangle upon a straight line* AB, *with the right angle opposite to the side* AB, (fig. 67.)

*Sol.* Bisect AB. From the division point C as a centre, with a radius equal to CA or CB, describe a semi-circumference. To AB at C erect a perpendicular, which will meet the semi-circumference at its middle point F. Draw FA and FB. FAB is the required triangle. If the proposition were to construct a right-angled triangle in general, we could take any point in the arc, and by connecting it by straight lines with A and B, should construct a right-angled triangle.

**77.** *To construct an obtuse-angled and an acute-angled isosceles triangle*, (fig. 67.)

*Sol.* Produce the perpendicular CF beyond the semi-circumference, and take in it the point D within, and the point E without the semi-circle. Draw DA, DB, EA, and EB. The triangle DAB is an obtuse-angled isosceles triangle, and the triangle EAB is an acute-angled isosceles triangle.

## CONSTRUCTION OF QUADRILATERALS.

**78.** *To construct a quadrilateral which shall be equal to a given quadrilateral* ABCD, (fig. 68.)

*Sol.* In the given quadrilateral ABCD draw the diagonal AC. It is thereby divided into 2 triangles, CAB and DAC. If we construct 2 triangles equal to these, and place them together in the same relative position, the problem will be solved.

Draw a straight line EF = AB. From E as a centre, with a radius = AC, and from F as a centre, with a radius = BC, describe 2 arcs cutting one another at G. Draw GE and GF. The triangle GEF = CAB.

Again, from E as a centre, with a radius = AD, and from G, with a radius = CD, describe 2 arcs cutting one another at H. Draw HE and HG. Triangle HEG = DAC. Consequently the quadrilateral EFGH = ABCD.

**79.** *To construct a square upon a given straight line* AB, (fig. 69.)

*Sol.* 1. At the points A and B erect perpendiculars to the line AB. Take AC and BD each equal to AB. Draw CD and the required square will be constructed.

*Sol.* 2. At the point A erect a perpendicular to the line AB. In this perpendicular take AC = AB. Through the point C draw CD equal and parallel to AB; and through B draw BD equal and parallel to AC. ABCD is the required square.

**80.** *To construct a square in and about a circle,* (fig. 70.)

*Sol.* Describe a circle. Draw 2 diameters cutting each other at right angles. Join the ends of these diameters by straight lines, and a square will be constructed within a circle.

At the points where the ends of the diameters meet the circumference, draw tangents to the circle. Produce each tangent both ways until they intersect one another; and a square will be constructed about a circle.

**81.** *To construct a square which shall be double a given square,* (fig. 72.)

*Sol.* In the given square ABCD, draw the diagonal AC, and construct upon it the square AEFC, which square will be double the square ABCD. For ABCD is composed of 2 equal triangles, ABC and ADC, but AEFC is composed of 4 triangles, each equal to ABC.

**82.** *Remark.* In a right triangle the side opposite to the right angle is called the *hypothenuse.* Now you will observe that the square constructed upon the hypothenuse AC of the isosceles right-angled triangle ABC is equal to the squares constructed upon AB and BC, the sides which include the right angle, taken together. We shall hereafter find this to be true in all right-angled triangles. //*

**83.** *To construct a square which shall be* 4, 9, 16, 25, 36, *&c. times as great as a given square,* (fig. 71.)

*Sol.* Let a square constructed upon AB be the given square. Produce AB, and take BC = AB, CF = AB, FI = AB, &c. Construct upon AC, AF, AI, &c., the squares ACDE, AFGH, AIKL, &c. ACDE is 4 times greater than the given square; AFGH is 9 times greater; and AIKL is 16 times greater.

If the side of a square is 2, its surface will be 4 times as great as that of a square whose side is 1; a square

* c υποτενουσα

whose side is 3 will have a surface 9 times as great as a square whose side is 1.

**84.** *To construct many rectangular and oblique-angled parallelograms upon a given straight line* AB, (fig. 73.)

*Sol.* At the point A erect a perpendicular to AB. Take any points C and D in this perpendicular; through them draw lines parallel to AB. Through B draw a line parallel to AC. Thus we have constructed 2 rectangular parallelograms upon AB.

Again, make at A an oblique angle. In the side which together with AB includes this angle take any points E and F, and through these points draw lines parallel to AB. Through the point B draw a line parallel to AE. We have thus constructed 2 oblique-angled parallelograms upon AB. If AC is taken equal to AB, the parallelogram will be a square; if we take AE = AB, the parallelogram will be a rhombus.

**85.** *Two adjacent sides* M *and* N *and the included angle* O *of a parallelogram being given, it is required to construct the parallelogram,* (fig. 74.)

*Sol.* Draw a straight line AB = M. At A make an angle BAD = O, and take AD = N. From D as a centre, with a radius equal to AB, describe an arc, and from the point B as a centre, with a radius equal to AD, describe another arc. From the point C, where these arcs cut each other, draw CD and CB. ABCD is the required parallelogram.

**86.** *The* 4 *sides and* 1 *angle of a quadrilateral being given, it is required to construct the quadrilateral,* (fig. 75.)

*Sol.* Let M, N, P and Q be the given sides, and O the angle included by M and N. Draw AB = M. At A make an angle CAB = O, and take AC = N. From the point C as a centre, with a radius equal to P, and from the point B as a centre, with a radius = Q, describe 2 arcs which will intersect at D. Draw DC and DB. ABCD is the quadrilateral required.

### CONSTRUCTION OF POLYGONS IN GENERAL.

**87.** *To construct a hexagon which shall be equal to a given hexagon* ABCDEF, (fig. 76.)

*Sol.* Divide the hexagon by diagonals into 4 triangles. Construct 4 triangles equal to those of the given hexagon and placed together in a similar order. The entire polygon GHMLKI thus constructed will be equal to ABCDEF.

In a similar manner polygons may be constructed, which shall be equal to a given polygon, whatever may be the number of its sides.

**88.** *To construct a regular polygon in and about a circle.*

*Sol.* Suppose the required polygon is an octagon. With the compasses divide the circumference of the circle into 8 equal parts. Connect the 8 division-points by chords; and at the same points draw tangents to the circle. In this manner one regular octagon will be constructed *within*, and another *without* the circle.

The problem can be solved by another mode. Construct a square in a circle. Bisect each of the sides of this square. Draw radii through these division points. Connect the ends of these radii by chords with the two

nearest vertices of the square. The required octagon will thus be constructed.

In a similar manner, by means of an inscribed equilateral triangle, we may construct a regular 6, 12, 24, &c. sided polygon in a circle, and by drawing tangents at the points where the vertices of the angles of such figures touch the circumference, we may construct a polygon of an equal number of sides about a circle.

## CONSTRUCTION OF CIRCLES.

**89.** *To describe a circle about a triangle.*

*Sol.* (Fig. 63. 1.) Bisect 2 sides of the triangle, and at each division-point erect perpendiculars, which will intersect each other at O. From O as a centre, with a radius equal to the distance from the point O to the vertex of one of the angles of the triangle, describe a circle. The circumference of this circle will pass through the vertices of all the angles of the triangle; it will therefore be the circle required.

*Remark.* If the triangle is right-angled, the centre of the circle will be in the middle of the hypothenuse; if the triangle is acute-angled, this centre will be within the triangle, and if it be obtuse-angled, it will be without the triangle.

**90.** *To describe a circle in a given triangle* ABC, (fig. 63. 2.)

*Sol.* Bisect the angles A and B by straight lines, which will intersect each other at O. From the point O let fall perpendiculars upon the 3 sides of the triangle. From O as a centre, with a radius equal to either of

these perpendiculars, describe a circle. The circumference of this circle will touch the 3 sides of the triangle. It will therefore be the circle required.

**91.** *To describe a circle in and about a given square.*

*Sol.* Draw 2 diagonals in the given square, and from the point where they intersect each other as a centre, with a radius equal to half a diagonal, describe a circle. The circumference of this circle will pass through the vertices of all the angles of the square, and thus we have a circle described about a square.

Again, from the point of intersection of the diagonals let fall a perpendicular upon one of the sides of the square; then, from the same point as a centre, with a radius equal to this perpendicular, describe a circle. It will be a circle inscribed in square.

**92.** *To describe a circle in and about a regular polygon.*

*Sol.* Bisect 2 adjacent sides of the polygon, and at the division points erect perpendiculars. From the point where these perpendiculars intersect each other as a centre, with a radius equal to one of the perpendiculars, describe a circle; it will be an inscribed circle.

Again, from this centre draw a line to the vertex of one of the angles of the polygon; and then with a radius equal to this line, describe a circle; it will be a circle circumscribed about the polygon.

## CONSTRUCTION OF THE SKELETONS OF SOLIDS.

**93.** We have before made rude diagrams of the skeletons of the solid bodies. We are now prepared to

construct them more accurately with the aid of instruments. The solid the skeleton of which is to be constructed should be placed before us. Construct the skeleton—

1. Of the cube, by placing together 6 equal squares, as shown in fig. 5.

2. Of the triangular, quadrangular, pentagonal, and polygonal prisms, as shown in figs. 1, 2, 3, 4.

3. Of the cylinder, fig. 6. The upper and lower sides of the rectangle must each be of the same length as the circumference of each circle.

4. Of the triangular pyramid, fig. 7.

5. Of the polygonal pyramids, figs. 7, 8, 9, and 10.

6. Of the cone, fig. 11. The curved side of the triangle must be of equal length with the circumference of the circle.

7. Of the regular solids, as shown in figs. 77, 78, 79, 80, 5.

# PART THIRD.

## COMPARISON AND MENSURATION.

### I. POINTS.

**94.** A *point* has no length, breadth, or thickness; it has in fact no extension; a point is not the smallest particle of a line. As a point has no extension it cannot be measured; one point is as large as another, or rather neither has any magnitude. The representation of a point on paper or on the board has a magnitude, else it would not be visible; but that which is represented has none. A point has only a position. Where a definite line, whether straight or curved, ends, there is a *point.* If two lines intersect, there is at the intersection a *point*, which lies in both lines. Place two points together, and the position of the one will not vary from the position of the other; they will have the same position, and will coincide.

**95.** If a point be moved, the path wnich it describes in moving will be a *line.* If the point moves forward in the same direction, it describes a *straight* line; if the direction be changed every moment, it describes a *curved* line; if the direction be changed only once, it describes one line, composed of two straight lines joined together, or a

*broken* line, the parts of which make an angle; if one point be moved round another point always at an equal distance, it describes a *circular* line. If a point remains in its course always at the same distance from a straight line, it will describe a straight line parallel to the former. If a point in a straight line be moved in a straight course, it must either continue in the same straight line, or it must leave the direction of that line. In this latter case, the straight line which it describes will make *an angle* with the former straight line. This angle may be either right or oblique.

## II. Lines.

**96.** *As a Line is the path described by the motion of a point*, it can only have extension in length; not in breadth or thickness. All lines have in common the property of extension in length; they may differ one from another in the quantity of the length, in their position, and in the position of their component parts. If these parts all lie in one and the same direction, the line is a straight line; if they do not all lie in the same direction, the line is either broken or curved. Three contiguous points in a curved line never lie in the same direction. Two points, let them lie as they may, can be connected by a straight line. A straight line cannot coincide with a curved line; they can only have one or more points in common.

**97.** A straight line is the shortest way from one point to another; every curved line between the same points is longer than the straight line. The more nearly the curved line approaches to a straight line, the shorter

it will be. There may be many curved lines between the same points; there can be only one straight line, because there can be only one shortest way. In geometry, when we speak of the distance between two points, we mean the length of a straight line, and therefore of the shortest line, which can connect them.

**98.** Two points determine exactly the direction of a straight line. Therefore if an engineer wishes to mark out a straight line in a field, he sets a stake at each end of the line, and then sets other stakes between these two, taking care that all shall be in the direction of the first two. This he ascertains by *taking sight* from one of these two to the other of them. If the straight line is to be made longer, he takes sight from one stake to another, and the stakes are successively set in the line of sight. In a similar manner a row of trees is set in a straight line, or a company of soldiers drawn up. Two straight lines which have 2 points in common must have the same direction, and must coincide.

### MENSURATION OF STRAIGHT LINES.

**99.** A magnitude can be measured only by comparing it with a magnitude of the same kind; thus, length can be measured only by length, surface by surface, weight by weight. Some one known magnitude is taken as the *unit*, and to measure a magnitude we seek how often this unit is contained in it. Therefore, *To measure a magnitude is to determine how often a known magnitude, which is taken as the unit of measure, is contained in the magnitude to be measured.* There are as many different *units of measure*, as there

are different kinds of magnitudes. In this school-room we might employ long, surface, and solid measures, since here are lengths, surfaces, and solids to be measured.

**100.** At present we will confine ourselves to the measure of length, or *long measure.* In measuring short lengths, we take as the unit of length, or *linear* unit, an *inch*, a *foot*, or a *yard.* When we actually perform the operation of measuring, we make use of a wooden or metallic *rule*, upon which the feet and inches are marked. If we have a line of great length to measure, we take a *rod* or a *mile* as the linear unit, and in performing the operation of measuring we make use of a wooden or metallic *rod*, of a *tape*, or of a *chain.* If a straight line is to be measured, we seek how often the unit of measure is contained in it. This may be done either by directly applying a rule, a rod, or a chain, or by another mode of which we shall have examples hereafter. If a curved line is to be measured, we seek how long it would be if it were extended in a straight direction. Thus a straight unit of measure is used to ascertain the length of all lines.

A chain is 5 rods long, = 22 yds. & is divided into 100 links being, 7.92 inch

## III. Of Angles.

### 1. Of Angles in General.

**101.** *An angle is the opening, or the mutual inclination of two lines meeting in a point.*

**102.** Angles in respect to the nature of their sides are divided into *rectilineal*, whose sides are straight

lines; *curvilineal*, whose sides are curved lines; *mixtilineal* having one side a curved line, the other a straight line. At present we have to do with rectilineal angles only.

### 2. Relative Magnitude.

**103.** Two angles may be either equal or unequal. They are equal when the sides of each have a similar inclination; in such case, if one be placed upon the other, they will entirely coincide; the vertex of one will coincide with the vertex of the other, and the sides of the one with the sides of the other. Equal angles agree in all respects, and the angles which do not agree in all respects are unequal.

### 3. Relative Position.

**104.** Two angles may lie entirely apart one from the other, so that their sides shall have no part in common.

*Or*, They may have one side entirely in common, or partly in common (fig. 81.)

The sides ~~the parts of~~ which are not ~~in~~ common may form one straight line, (fig. 82.) Such angles are called *adjacent angles*, as angles R and S.

*Or*, Two angles may have only the vertical point in common. If such angles are equal, and the sides of one being produced coincide with the sides of the other, thus forming 2 straight lines, the angles are called *vertical* angles; as the angles O and X; M and N, (fig. 83.)

**105.** If two parallel straight lines are crossed by a third line, the angles thus made have a relative position

one to another, which is expressed by particular names. Thus, (fig. 84,)

C and F, D and E, &c., are *alternate internal angles*, because they are on opposite sides of the single line, and within the parallels.

A and H, B and G, &c., are *alternate external angles.*

D and F, C and E, &c., are *interior angles upon the same side*, because they are contained between the parallels, and are on the same side of the line which crosses them.

A and G, B and H, &c., are *exterior angles upon the same side.*

A and E, B and F, G and C, H and D, are *external internal angles*, because one is without and the other within the parallels, and both are on the same side of the single line.

We have also many pairs of adjacent angles; viz., A and B, A and C, B and D, E and G, &c.; and many pairs of vertical angles, viz., A and D, B and C, E and H, &c.

### 4. Magnitude of Angles taken together.

**106.** *Two adjacent angles are together equal to* 2 R. A. *or* 180°. This follows directly from (**34.**) Therefore, if one of the angles $= \frac{1}{3}$ R. A., then the other $= 2$ R. A. $- \frac{1}{3}$ R. A. $= 1\frac{2}{3}$ R. A. The one $= 30°$, the other $= 150°$.

All the angles in the same plane about a point are together equal to 4 R. A., or 360°. Hence, when one line crosses another, since all right angles are equal, if one of the angles is a right angle, then all are R. A.

**107.** *Two vertical angles are equal one to the other*, (fig. 83.) M = N, O = X.

*Dem.* $M + O = 2$ R. A. and $M + X = 2$ R. A. (**106.**) Therefore $M + O = M + X$; for it is an axiom that *things which are equal to the same thing are equal to one another*. $M = M$, therefore $O = X$, for it is also an axiom, that *if equals be taken from equals the remainders will be equal.* In a similar manner it may be demonstrated that $M = N$; and in general, that the vertical angles made by any number of straight lines crossing at one point are equal to each other.

**108.** *Two external-internal angles are equal one to the other*, (fig. 84.) $A = E$, $B = F$, &c.

For two parallel lines have a similar position, one with the other; therefore a straight line crossing them has the same inclination to each, that is, it makes equal angles with each.

**109.** *The two interior angles upon the same side taken together are equal to* 2 R. A., (fig. 84.) $D + F =$ 2 R. A.; $C + E = 2$ R. A.

*Dem.* $B = F$ (**108**) and $D = D$. It is an axiom that *if equals be added to equals the sums will be equal;* therefore $B + D = F + D$. But $B + D = 2$ R. A., (**106**); therefore $F + D = 2$ R. A.

**110.** *Two alternate-internal angles are equal one to the other*, (fig. 84.) $C = F$, $D = E$, &c.

*Dem.* $C + E = 2$ R. A. (**109**), and $F + E = 2$ R. A. (**106**); therefore $C + E = F + E$; therefore $C = F$.

**111.** It may also be demonstrated that the *alternate-external* angles are equal to each other, (for example $A = H$; $B = G$;) that *two alternate-internal-external* angles taken together are equal to 2 R. A.; (that is, $A +$

F = 2 R. A., B + E = 2 R. A.;) and that ***two exterior angles upon the same side*** are together equal to 2 R. A. (that is, A + G = 2 R. A.; B + H = 2 R. A.)

**112.** Reciprocally, we may infer that two straight lines are parallel, if, when crossed by a third straight line, the following propositions are true; and the reverse of this, that two straight lines are not parallel, if these propositions are not true, viz., if,

The external-internal angles are equal.

The alternate-internal angles are equal.

The alternate-external angles are equal.

The interior angles upon the same side are together equal to 2 R. A.

The exterior angles upon the same side are together equal to 2 R. A.

The alternate-external-internal angles are together equal to 2 R. A.

**113.** *If two straight lines* AB *and* CD *are each parallel to a third line* EF, *they are parallel to each other* (fig. 85.)

*Dem.* Draw a straight line crossing the other three. Then, because AB is parallel to EF, therefore angle O = Y (**108**), and because CD is parallel to EF, therefore angle X = Y; therefore O = X, consequently AB is parallel to CD.

**114.** *Two straight lines are parallel to each other, if each is perpendicular to a third line;* for the interior angles upon the same side taken together will be equal to 2 R. A.

### 5. Mensuration of Angles.

**115.** For the measure of angles upon paper we make use of a protractor. For measuring angles in the field various instruments are used, more or less complicated. The most simple is a graduated circle, (which may be made of metal, wood, or pasteboard,) with an index moving on a pivot in the middle. Place the instrument at the vertex of the angle to be measured, and make the index coincide in direction with one of the sides of the angle; then move the index until it coincides in direction with the other side of the angle, noting the number of degrees on the graduated circle which it passes over; this number will be the magnitude of the angle.

For determining directions which shall be perpendicular to the surface of still water, that is, to a horizontal surface, a leaden weight hanging freely from a string, and called a *plumb*, is used. Positions which are parallel to the direction of the plumb, are perpendicular to the horizon, that is *vertical.* Masons use the plumb in building walls.

For determining *horizontal positions* a *plumb level*, (fig. 88,) is frequently used. It consists of a wooden frame made in the form of an isosceles triangle, with a plumb attached to it. The base of the triangle is placed upon the surface the position of which is to be determined; if the plumb falls directly over the marked centre point of the base, then this base, and consequently the surface upon which it stands, is horizontal. This kind of *level* is sometimes made in the form of the letter L. The two parts must be exactly perpendicular, one to the other; a plumb is suspended from the top

of the vertical ruler; and if the string coincides exactly with the edge of this ruler, then the other ruler must be horizontal. This kind of level is also made in the form of an inverted T. viz., ⊥.

But the *spirit level* is the most accurate, and the one most commonly used. It consists of a cylindrical glass tube filled with spirits of wine, excepting a small portion containing air; the ends of the tube being hermetically sealed. The bubble of the air, being the lightest part of the contents of the tube, will always run towards that end which is highest; but when the tube is horizontal it will have no tendency to either end. The *bore* of the tube is not exactly cylindrical, but it is slightly curved, the convex side being upward; therefore the bubble will rest in the middle of the tube, when the tube is horizontal. The tube is fitted into a block of wood, the bottom of which is exactly parallel to the tube; so that when the bottom of the block is horizontal, the bubble will be exactly between two scratches marked on the top of the tube to show the middle.

## IV. Figures.

### 1. Figures in General.

**116.** We defined a line to be the path described by the motion of a point. In a similar manner we may say that *a surface is the space described by a line moved in any direction but that of its length.* For example, if you suppose a straight line to be turned on a pivot at one extremity or in the middle, it will describe a circle; if it be moved in a direction perpendicular to its length it will describe a parallelogram. If we suppose a curved line to revolve on an axis connecting its extremities it

will describe a curved surface. A line has no thickness, consequently a surface has no thickness. A line has length, and its motion makes breadth, consequently a surface has extension in two directions, viz., length and breadth.

### 2. Triangles in General.

**117.** *In every triangle one side is less than the sum of the other two.* For a straight line is the shortest distance between two points; therefore (fig. 87) $AB < AC + CB$.

**118.** *In every triangle the* 3 *angles are together equal to two right angles.*

*Dem.* Through the vertex of the angle A, (fig. 87,) draw a straight line parallel to the opposite side BC. Then $O = B$, and $X = C$ (**110**). Consequently $O + X + A = B + C + A$. But $O + X + A = 2$ R. A. (**34**); therefore $B + C + A = 2$ R. A.

**119.** A *corollary* is a consequence which follows directly from a proposition. From the preceding proposition we have the following corollaries.

**120.** *Cor.* 1. In every triangle there can be but 1 R. A., and but 1 obtuse angle; there must be at least 2 acute angles.

**121.** *Cor.* 2. In every right triangle the 2 acute angles are together equal to 90°. In every obtuse-angled triangle the 2 acute angles are together less than 90°.

**122.** *Cor.* 3. If 2 angles of a triangle are equal, each is an acute angle.

**123.** *Cor.* 4. If 2 angles of a triangle are given, the 3d angle may easily be found, for it is equal to 2 R. A. less the sum of the given angles. If one angle $=64°$, and another $=70°$, then the 3d angle $=180°-64-70°=46°$. Thus if two angles of one triangle are equal to two angles of another, then the third angle of the one must be equal to the third angle of the other.

**124.** *The exterior angle of a triangle, that is, the angle made by producing one of the sides of a triangle, is equal to the sum of the two opposite interior angles,* (fig. 88.) Angle ACD = angle ABC + angle CAB.

*Dem.* ACD + ACB = 2 R. A.; ABC + CAB + ACB = 2 R. A.; therefore ACD + ACB = ABC + CAB + ACB. Subtract ACB from each side, and we have ACD = ABC + CAB.

**125.** *The three exterior angles of a triangle are together equal to* 4 R. A. For the sum of each pair of adjacent angles is 2 R. A.; in all $3\times2=6$ R. A. The interior angles together = 2 R. A., therefore the exterior angles together $=6-2=4$ R. A.

### 3. Of Equal Triangles.

**126.** *Two triangles are equal, if 2 sides and the included angle of the one are equal to 2 sides and the included angle of the other, each to each.*

*Dem.* Let BC = EF, BA = ED, and angle B = E, (fig. 62.) Now suppose the triangle ABC to be placed upon DEF, so that point B shall fall upon E, and C

upon F; this is possible, because BC = EF; in this case BC will coincide with EF. Because angle B = E, BA must take the direction ED, and because BA = ED, the point A must fall on D. Thus the end-points C and A of the line CA fall upon the points F and D, which are the ends of the straight line FD, consequently CA and FD must coincide. Thus the two triangles coincide throughout, and consequently they are equal.

**127.** *Two triangles are equal if two angles and the interjacent* (lying between) *side of the one are equal to two angles and the interjacent side of the other, each to each.*

*Dem.* (fig. 62.) Let angle B = E, angle C = F, and side BC = EF. Suppose the triangle ABC to be laid upon DEF, so that BC shall coincide with EF. Because angle B = E, BA must be in the direction of ED, and the point A must fall in ED; because angle C = F, CA must be in the direction of FD, and point A must fall in FD. Thus the point A is in both the lines FD and ED; it can only be at their point of intersection D. Consequently the two triangles coincide throughout, and are equal.

**128.** *Remark.* In both these cases of equal triangles, equal angles are opposite to equal sides.

**129.** *In every isosceles triangle the angles opposite to the equal sides are equal,* (fig. 86.)

If ACB is an isosceles triangle of which the sides AC and CB are equal, then the angles A and B are likewise equal.

*Dem.* Bisect the third angle C by a straight line which shall meet the opposite side at the point D. In

the triangles ACD and BCD, the sides AC and CB are equal by supposition, CD is common, and the angle ACD = BCD by construction. Therefore triangle ACD = BCD, (**126**,) consequently angles A and B are equal.

**130.** *Cor.* 1. *If the two equal sides of an isosceles triangle be produced, the angles formed without the triangle will be equal*, for each of them, together with one of the equal angles A and B, is equal to 2 R. A. (**107.**)

**131.** *Cor.* 2. Equilateral triangles are also equiangular; and each angle is equal to 60°.

**132.** *Cor.* 3. If one angle of an isosceles triangle be given, the others may easily be found. Let one of the angles at the base be given; because the other angle at the base is equal to this, the sum of the two subtracted from 180° will give the third angle. If the angle opposite to the base is given, subtract it from 180°, divide the remainder by 2, and the quotient will be the size of each angle at the base.

The *base* of an isosceles triangle is the side which is not equal to one of the others. In triangles not isosceles either side may be taken for the base. The vertex of the angle opposite to the base is sometimes called the *vertex* of the triangle.

Figure 86

**133.** From the equality of the triangles ACD and BCD the following propositions may be deduced.

1. The straight line which bisects the angle C is perpendicular to the base AB. Because the adjacent angles CDA and CDB are equal and therefore right angles.

2. A straight line drawn from the vertex of an isos-

$\Delta$ are



celes triangle perpendicular to the base, bisects that base.

3. If a perpendicular erected upon the middle point of one side of a triangle meets the vertex of the opposite angle, then the other two sides are equal to each other.

4. A perpendicular erected upon the middle point of the base of an isosceles triangle, bisects the opposite angle.

**134.** *If two angles of a triangle are equal (the sides opposite to these angles are also equal.)* The tri ang. is isosceles

*Dem.* (fig. 86.) Let angle A = B. Upon the middle point of the base AB erect a perpendicular to AB. If this perpendicular passes through the vertex of the triangle, then AC = BC, (**133.** 3.) Suppose it does not pass through the vertex, but meets the side AC at E. Draw EB. Triangles EAD and EBD are equal, (**126**;) consequently angle EAD = EBD. But angle EAD = CBD; consequently angle EBD = CBD; that is, a part is equal to the whole, which is absurd. Therefore, the perpendicular CD does pass through the vertex of the triangle, and the sides AC and BC are equal.

**135.** *Two right angled triangles are equal, if two sides of the one are equal to two sides of the other, each to each.*

*Dem.* If the equal sides include the right angles, then the equality of the triangles has already been demonstrated, (**126.**)

But let us suppose the hypothenuse AB and side BC of triangle ABC (fig. 89. 2.) to be respectively equal to the hypothenuse DE and side EF of triangle DEF. Suppose the triangle ABC to be placed so that BC shall coincide with EF. AB will take the direction of EG, and, because the angles DFE and ACB are right angles,

AC will coincide with DF produced; suppose with FG. We shall thus have an isosceles triangle DEG. Now EF is perpendicular to DG, and therefore bisects it, (**133.** 2;) thus GF = AC = DF. Therefore the triangles ABC and DEF are equal, (**126.**)

**136.** We are now prepared to demonstrate another principle, viz., that any point in a perpendicular EC (fig. 67,) erected at the middle of a straight line AB, is at equal distances from the two extremities of this line. Let us take the point D, and draw the lines DA and DB. The triangles DAC and DCB are equal; for AC and CB are equal by supposition, and CD is common to both. Therefore the sides AD and DB are equal, which was to be demonstrated.

**137.** *Cor.* Two oblique lines, as AD and DB, drawn at equal distances from a perpendicular, are equal.

**138.** *Two triangles are equal if three sides of the one are equal to three sides of the other, each to each,* (fig. 89. 1.) If AB = DE, AC = DF, and BC = EF, then is triangle ABC = DEF.

*Dem.* Suppose the triangle ABC to be moved from its place so that BC shall coincide with EF, and the triangle ABC shall occupy the space GEF; BA coinciding with EG, and CA with FG. Draw DG. The angles C and F may be either acute, right, or obtuse, and each case will be considered separately.

1. Acute angles, (fig. 89. 1.) By supposition ED = EG and FD = FG, therefore angle EDG = EGD, and angle FDG = FGD, (**129**;) consequently angle EDG + FDG = EDF = angle EGD + DGF = EGF. Now

angle EGF = BAC, therefore angle EDF = BAC. Consequently triangle ABC = triangle DEF, (**126.**)

**2.** Right angles, (fig. 89. 2.) By supposition ED = EG, therefore angle D = G. Angle A = G, therefore A = D; consequently triangle ABC = triangle DEF, (**126.**)

3. Obtuse angles, (fig. 89. 3.) By supposition ED = EG and FD = FG, therefore angle EDG = EGD, and angle FDG = FGD. Consequently angle EDG — FDG = angle EDF = EGD — FGD = EGF. Now angle BAC = EGF, therefore angle BAC = EDF; and the triangle ABC = DEF, (**126.**)

In each case equal angles are opposite to equal sides.

**139.** *In every triangle the greater side is opposite to the greater angle*, (fig. 90.) If AC > AB, then is angle B > angle C.

*Dem.* In the side AC take AD = AB and draw BD. Angle ABD = ADB, (**129.**) But angle ADB > ACB, (**124**,) much more is angle ABC = ABD + DBC > ACB.

**140.** *Cor.* In every triangle whose sides are not equal, the greatest side is opposite to the greatest angle, and the least side to the least angle.

**141.** *In every triangle the greatest angle is opposite to the greatest side*, (fig. 90.) If angle B > C, then is AC > AB.

*Dem.* Draw BD, making the angle CBD = angle C. Because angle CBD = BCD, therefore BD = CD, (**133**;) DA = DA; and therefore BD + DA = CD + DA. But BD + DA > BA, consequently CD + DA = CA > BA.

**142.** In every triangle whose angles are not equal, the least angle is opposite to the least side, the mean angle to the mean side, and the greatest angle to the greatest side.

**143.** *In the two triangles* ABC *and* DEF, (fig. 109,) *let the sides* AB *and* AC *be respectively equal to the sides* DE *and* DF, *but the angle* EDF *be greater than the angle* BAC; *then will the side* EF *be greater than the side* BC.

*Dem.* Let the triangle ABC be placed upon the triangle DEF, so that AB shall coincide with DE; but as angle BAC is less than EDF, the side AC will not fall upon DF, but will fall within it, suppose in the position DG; BC will lie in the position EG, and triangle DEG will be equal to triangle ABC. Draw FG. In the isosceles triangle DFG, angle DGF = angle DFG, (**129.**) It is self-evident that DFG is greater than EFG; therefore DGF is greater than EFG; much more is EGF greater than EFG; therefore the side EF is
141 greater than the side EG, or its equal BC, (**142.**)

**144.** The converse of this proposition is true, viz.: that if the sides AB and AC are respectively equal to the sides DE and DF, but the side EF is greater than the side BC, then will the angle EDF be greater than the angle BAC. For angle EDF cannot be equal to angle BAC, for then the two triangles will be equal, and the side EF = BC, which is contrary to the supposition; neither can angle EDF be less than angle BAC, for then the side EF would be less than the side BC, which is also contrary to the supposition; angle EDF must therefore be greater than angle BAC.

### 4. Quadrilaterals.

**145.** *In every quadrilateral the sum of all the angles is equal to* 4 R. A.

*Dem.* By a diagonal the quadrilateral is divided into 2 triangles. The sum of the angles of each triangle is equal to 2 R. A.; consequently the sum of the angles of both triangles, that is, of the quadrilateral, is equal to 4 R. A.

**146.** *Every parallelogram* ABCD *is divided by a diagonal* DB *into two equal triangles*, (fig. 91.)

*Dem.* AB is parallel to DC, therefore angle ABD =BDC, (**110**;) AD is parallel to BC, therefore angle BDA=DBC; BD=BD; therefore triangle ABD= CBD, (**127.**)

**147.** *Cor.* In equal triangles equal sides are opposite to equal angles; consequently AD=CB, and AB =CD; that is, in every parallelogram the sides opposite to each other are equal. Each part of the angle B is equal to a part of the angle D, therefore the entire angle B is equal to the entire angle D; the angle C=A; that is, in every parallelogram the opposite angles are equal.

**148.** *Equivalent* figures are such as have equal surfaces. Two figures may be equivalent, although dissimilar in form; thus a circle may be equivalent to a square.

**149.** The *altitude* of a parallelogram is a line drawn from one side perpendicular to the opposite side considered as the base, or to that side produced if necessary. The *altitude* of a triangle is a perpendicular

drawn from the vertex of an angle to the opposite side taken as a base; as CD is the altitude of triangle ABC, (fig. 86.)

**150.** *Two parallelograms having the same or equal bases and an equal altitude, are equivalent.*

*Dem.* Upon one of the sides of the parallelogram ABCD (fig. 92,) construct a second parallelogram ABEF, of which the side opposite to AB shall be in DC produced, that is, which shall have an equal altitude with ABCD. We are to prove that ABEF is equivalent to ABCD. AB = DC; AB = FE, (**147**,) therefore DC = FE; CF = CF; therefore DC + CF = DF = FE + CF = EC. Again, AD = BC, and AF = BE, therefore triangle ADF = BCE, (**138.**)

Triangle OCF = OCF, therefore triangle ADF — OCF = ADCO is equal to triangle BCE — OCF = BOFE. Triangle AOB = AOB, therefore ADCO + AOB = ABCD = BOFE + AOB = ABEF.

Let the parallelograms have not the same but equal bases, and have equal altitudes; we can suppose one to be placed upon the other, so that the bases shall coincide, and then the case becomes identical with the preceding.

**151.** *Cor.* Every oblique parallelogram is equivalent to a right parallelogram of an equal base and an equal altitude.

**152.** *Every triangle is equivalent to one half of a parallelogram of equal base and equal altitude.*

*Dem.* Let BC = EF, and AG be parallel to BF. Draw CH parallel to BA. Parallelograms ABCH and

DEFG are equivalent, (**150.**) Triangle ABC $= \frac{1}{2}$ ABCH, therefore it is equivalent to $\frac{1}{2}$ DEFG.

**153.** *Cor.* Two triangles of equal bases and equal altitudes are equivalent, because each is the half of a parallelogram having an equal base and an equal altitude.

### 5. Polygons in General.

**154.** *The sum of all the angles of a rectilineal figure is equal to twice as many right angles, wanting 4, as the figure has sides.*

*Dem.* From any point within the figure draw straight lines to the vertex of each angle. Thus the figure will be divided into as many triangles as it has sides. The sum of all the angles of each triangle $= 2$ R. A., therefore the sum of all the angles of all the triangles is equal to 2 R. A. multiplied by the number of sides in the figure. But the angles (equal to 4 R. A.) about the point within the figure do not belong to the figure; therefore we must deduct 4 R. A.

For example, the angles of a

| | | |
|---|---|---|
| triangle together | $= (3 \times 2) - 4 =$ | 2 R. A. |
| quadrilateral | $= (4 \times 2) - 4 =$ | 4 R. A. |
| pentagon | $= (5 \times 2) - 4 =$ | 6 R. A. |
| 20n | $= (20 \times 2) - 4 =$ | 36 R. A. |

**155.** In a regular polygon all the angles are equal, therefore each angle of a regular triangle

$= \frac{1}{3} \times 2$ R. A. $= \frac{2}{3}$ R. A. $= \frac{2}{3} \times 90° = 60°$

Of a regular quadrilateral,

$= \frac{1}{4} \times 4$ R. A. $=$ 1 R. A. $= 1 \times 90° = 90°$

Of a regular hexagon,

$= \frac{1}{6} \times 8$ R. A. $= \frac{8}{6}$ R. A. $= 1\frac{1}{3} \times 90° = 120°$

Of a regular 20n, Eikosagon

$= \frac{1}{20} \times 36$ R. A. $= \frac{36}{20}$ R. A. $= 1\frac{4}{5} \times 90° = 162°$

Of a regular 1000n, [illegible]

$= \frac{1}{1000} \times 1996$ R. A. $= \frac{1996}{1000}$ R. A. $= 1\frac{996}{1000} \times 90° = 179\frac{640}{1000}°$

**156.** Suppose a circle to be circumscribed about a regular polygon, and a straight line to be drawn from its centre to each angular point of the polygon; there will thus be formed about the centre as many equal angles as the polygon has sides.

If the polygon be a

| | | | | |
|---|---|---|---|---|
| triangle each centre angle | | $=$ | $\frac{4}{3}$ R. A. $=$ | $120°$ |
| quadrilateral | " " | $=$ | $\frac{4}{4}$ R. A. $=$ | $90°$ |
| hexagon | " " | $=$ | $\frac{4}{6}$ R. A. $=$ | $60°$ |
| decagon | " " | $=$ | $\frac{4}{10}$ R. A. $=$ | $36°$ |

The greater the number of the sides of the inscribed polygon, the greater will be its angles, but the less will be the centre angles. One centre angle, and one angle of the inscribed polygon are together equal to 2 R. A.; because they are equal to the three angles of a triangle. For example, each angle of a regular hexagon $= 120°$; each centre angle $= 60°$; and $60° + 120° = 180°$.

**157.** The angles made by producing the sides of a polygon are called exterior angles. Each exterior angle with its adjacent interior angle are together $=$ 2 R. A. Therefore the sum of all the exterior and interior angles is equal to as many times 2 R. A. as the polygon has sides. But the sum of all the interior angles is equal to as many times 2 R. A. as the figure has sides, less 4 R.

A.; consequently, *the sum of all the exterior angles of a polygon is equal to* 4 R. A.

**158.** The size of the interior angles increases with the number of sides, consequently that of the exterior angles must decrease in the same proportion. The size of each exterior angle of a regular polygon may be found by dividing 4 R. A. by the number of sides in the polygon. Thus each exterior angle of a triangle $= \frac{4}{3}$ R. A. $= 120°$

" " pentagon $= \frac{4}{5}$ R. A. $= 72°$ &c.

## TABLE OF THE SIZE OF ANGLES IN A REGULAR FIGURE.

| Number of the Sides. | Sum of the Interior Angles. | Size of each Interior Angle. | Size of each Exterior Angle. | Size of each Angle at the centre of a circle circumscribed about a regular figure. |
|---|---|---|---|---|
| 3 | 2 R. A. | $\frac{2}{3}$ R. A. = 60° | $\frac{4}{3}$ R. A. = 120° | $\frac{4}{3}$ R. A. = 120° |
| 4 | 4 R. A. | $\frac{4}{4}$ R. A. = 90° | $\frac{4}{4}$ R. A. = 90° | $\frac{4}{4}$ R. A. = 90° |
| 5 | 6 R. A. | $\frac{6}{5}$ R. A. = 108° | $\frac{4}{5}$ R. A. = 72° | $\frac{4}{5}$ R. A. = 72° |
| 6 | 8 R. A. | $\frac{8}{6}$ R. A. = 120° | $\frac{4}{6}$ R. A. = 60° | $\frac{4}{6}$ R. A. = 60° |
| 7 | 10 R. A. | $\frac{10}{7}$ R. A. = $128\frac{4}{7}$° | $\frac{4}{7}$ R. A. = $51\frac{3}{7}$° | $\frac{4}{7}$ R. A. = $51\frac{3}{7}$° |
| 8 | 12 R. A. | $\frac{12}{8}$ R. A. = 135° | $\frac{4}{8}$ R. A. = 45° | $\frac{4}{8}$ R. A. = 45° |
| 9 | 14 R. A. | $\frac{14}{9}$ R. A. = 140° | $\frac{4}{9}$ R. A. = 40° | $\frac{4}{9}$ R. A. = 40° |
| 10 | 16 R. A. | $\frac{16}{10}$ R. A. = 144° | $\frac{4}{10}$ R. A. = 36° | $\frac{4}{10}$ R. A. = 36° |
| 11 | 18 R. A. | $\frac{18}{11}$ R. A. = $147\frac{3}{11}$° | $\frac{4}{11}$ R. A. = $32\frac{8}{11}$° | $\frac{4}{11}$ R. A. = $32\frac{8}{11}$° |
| 12 | 20 R. A. | $\frac{20}{12}$ R. A. = 150° | $\frac{4}{12}$ R. A. = 30° | $\frac{4}{12}$ R. A. = 30° |
| 90 | 176 R. A. | $\frac{176}{90}$ R. A. = 176° | $\frac{4}{90}$ R. A. = 4° | $\frac{4}{90}$ R. A. = 4° |
| 360 | 716 R. A. | $\frac{716}{360}$ R. A. = 179° | $\frac{4}{360}$ R. A. = 1° | $\frac{4}{360}$ R. A. = 1° |
| 21600 | 43196 R. A. | $\frac{43196}{21600}$ R. A. = 179°. 59′. | $\frac{4}{21600}$ R. A. = 1′ | $\frac{4}{21600}$ R. A. = 1′ |
| 1296000 | 2591996 R. A. | $\frac{2591996}{1296000}$ R. A. = 179°. 59′. 59″. | $\frac{4}{1296000}$ R. A. = 1″ | $\frac{4}{1296000}$ R. A. = 1″ |

In a regular polygon having an infinite number of sides, (and a circle may be supposed to be such a polygon,) each interior angle = 180°, and each exterior angle = 0°.

### 6. Mensuration of Rectilineal Figures.

**159.** A quantity is measured by comparing it with some known quantity of the same kind, (**9~~8~~.**) The 9/ most simple of surfaces is a square, for in this figure the two dimensions are the same. The *square*, therefore, has been adopted as the unit of measure for surfaces, or *superficial unit;* and its side is taken equal to some linear unit, as an inch, a foot, a mile, &c. To measure a surface we seek how often the unit of measure is contained in that surface, that is into how many squares, of equal magnitude with the unit, the surface to be measured can be divided. *Area* and *surface* are nearly synonymous terms. But *area* is more particularly a surface measured, and expressed in numbers of superficial units.

**160.** Here we have a right parallelogram ABCD, (fig. 94.) Let the side AB contain 5 linear units, and the side AD contain 3 of the same units; and let the sides be divided accordingly. From each division-point of AB draw a line parallel to AD; and from each division-point of AD draw a line parallel to AB. The surface of the parallelogram will thus be divided into $5 \times 3$ equal squares; each of the sides of which is equal to the linear unit. If the linear unit be 1 inch, then each square is 1 square inch, and the area of the parallelogram is $5 \times 3 = 15$ square inches. If the linear unit is a rod, then each square is a square rod, and the parallelogram contains 15 square rods. Thus we learn that *the area of a right parallelogram is found by multiplying together two adjacent sides, that is, the base by the altitude.*

But the oblique parallelogram is equal in surface to a

right parallelogram of equal base and equal altitude, (**151**;) consequently *the area of any parallelogram may be found by multiplying its base by its altitude.*

**161.** Every triangle is equivalent to the half of a parallelogram of equal base and equal altitude. Hence *the area of a triangle is found by multiplying its base by its altitude, and dividing the product by* 2. We shall come to the same result by multiplying the whole base by $\frac{1}{2}$ the altitude, or the whole altitude by $\frac{1}{2}$ the base. Thus the area of triangle ABC (fig. 86) $= \frac{AB \times CD}{2} = \frac{1}{2} AB \times CD = AB \times \frac{1}{2} CD$.

**162.** To find the area of polygons in general we must divide them into triangles, and then find the area of each triangle by itself; the sum of the areas of all the triangles will be the area of the polygon. In the pentagon ABCDE, (fig. 96,) draw the diagonals AD and AC. From the vertex of the angle B let fall upon AC the perpendicular BF. From the vertices of the angles C and E let fall upon opposite sides of AD the perpendiculars CG and EH. The area of triangle ABC $= \frac{AC \times BF}{2}$; triangle ACD $= \frac{AD \times CG}{2}$; triangle ADE $= \frac{AD \times EH}{2}$; therefore the area of the pentagon ABCDE $=$

$$\frac{(AC \times BF) + (AD \times CG) + (AD \times EH.)}{2}$$

**163.** The area of the trapezoid ABDC, (fig. 95) is equal to $(CF \times \frac{1}{2} AB) + (BE \times \frac{1}{2} CD.)$ $CF = BE$, therefore the area of ABDC is equal to $CF \times (\frac{1}{2} AB + \frac{1}{2} CD;)$ that is, *the area of a trapezoid is found by multi-*

*plying half the sum of its parallel sides by the distance between those sides.*

**164.** *The area of any regular polygon may be found by multiplying its perimeter by half the radius of the inscribed circle.*

Take for example the regular pentagon ABCDE, (fig. 97.) From the centre O of the inscribed circle draw the lines OA, OB, &c., to the vertices of all the angles. As many triangles will thus be formed as the figure has sides. The area of the polygon is equal to the sum of the products of the bases of these triangles, viz., AB, BC, &c., by half their altitudes OF, OG, &c. But these altitudes are radii of the inscribed circle, and therefore are equal; therefore we can multiply the sum of the bases, that is the perimeter of the polygon, by $\frac{1}{2}$ the altitude of one triangle, that is, one half the radius of the inscribed circle.

## 7. The Circle.

**165.** The length of the radius determines the distance of the circumference of a circle from the centre, and consequently determines the size of the circle. Therefore, *circles having equal radii or equal diameters are equal.*

**166.** *In a circle, or in equal circles, equal angles, the vertices of which are at the centre, will intercept equal arcs upon the circumference, and the chords which subtend those arcs will also be equal; conversely, equal chords subtend equal arcs, and the angles at the centre measured by those arcs are equal.*

first time



If the arc AB = arc CD, then the chord AB = chord CD, (fig. 98.)

*Dem.* Draw the radii OA, OB, OC, and OD. Because the angles at the centre are measured by the arcs intercepted by their legs, (**59**,) equal arcs must be intercepted by equal angles; thus the angle AOB = COD; OA = OC, and OB = OD by construction; consequently triangle AOB = COD (**126**;) therefore AB = CD.

Conversely; AB = CD, OA = OC, OB = OD, consequently triangle COD = AOB, (**138**;) therefore angle AOB = COD. Again, the arcs AB and CD belong to the same circle, and therefore have the same curvature; the chord AB = chord CD, by supposition, therefore if the segment AB be laid upon segment CD, so that chord AB shall coincide with chord CD, then the arc AB must coincide with arc CD; therefore they are equal.

**167.** *A straight line drawn from the centre of a circle to the middle of a chord in the same circle is perpendicular to that chord, and bisects the arc subtended by that chord, and the angle at the centre measured by the arc.*

*Dem.* (Fig. 99.) Draw the radii OA and OB. Now AC = CB by supposition, OA = OB by construction, and OC = OC; consequently triangle AOC = BOC, (**138**;) therefore the adjacent angles OCA and OCB are equal, and therefore right angles; consequently OC is perpendicular to AB. The angles COA and COB are likewise equal; therefore arc AD = arc DB; consequently the angle O and the arc AB are bisected.

**168.** *A straight line drawn from the centre of a circle perpendicular to a chord of the same circle bisects that chord.*

*Dem.* (Fig. 99.) Draw the radii OA and OB. We have now an isosceles triangle of which the centre O is the vertex, and the chord AB is the base. A perpendicular let fall from O upon AB must bisect AB, (**133.**)

**169.** *Cor. A perpendicular erected upon the middle of a chord passes through the centre of the circle.* Therefore we can find the centre of a circle by erecting a perpendicular upon the middle point of each of two chords; the point where these perpendiculars intersect each other will be the centre of the circle.

**170.** *The perpendicular* AB *erected at the extremity of* OA, *a radius of a circle, is a tangent to the same circle,* (fig. 101.)

*Dem.* In AB take any point C. Draw OC. Angle OAC is a right angle by supposition, therefore angle OCA < 1 R. A. (**120**;) therefore OC > OA (**1[illegible]**;) But OA is a radius, consequently the point C, and every point in the line AB is more than the length of a radius distant from the centre O, and therefore is without the circle. Thus the line AB touches the circumference only at one point, and therefore is a tangent to the circle.

**171.** *Cor.* A ~~radius or~~ a diameter drawn from the point of contact of a tangent to a circle is perpendicular to that tangent; otherwise there might be two perpendiculars to a straight line at one point; that is, two straight lines which shall meet it so as to make the adjacent angles equal; which is impossible.

**172.** An *inscribed* angle is one made by 2 chords and having its vertex in the circumference. An angle

inscribed in a segment is one made by 2 chords drawn from any point in the arc of the segment to the extremities of its chord or *base*.

*An inscribed angle* ACB, (fig. 102,) *has for its measure half of the arc* AB *intercepted between its sides.*

*Dem.* The centre of the circle, O, may be in one of the sides; between the sides; or without the sides.

*Case* 1. (Fig. 102. 1.) Angle $AOB = OCB + OBC$, (**124.**) $OC = OB$, consequently angle $OCB = OBC$, (**129**;) therefore angle $AOB = 2\ OCB = 2\ ACB$.

*Case* 2. (Fig. 102. 2.) Draw the diameter CD. Angle $AOD = 2\ ACD$; angle $BOD = 2\ BCD$; therefore angle $AOD + BOD\ (= AOB) = 2\ ACD + 2\ BCD\ (= 2\ ACB.)$

*Case* 3. (Fig. 102. ~~4~~ 3.) Draw the diameter CD. Angle $DOB = 2\ DCB$; angle $DOA = 2\ DCA$; therefore angle $DOB - DOA\ (= AOB) = 2\ DCB - 2\ DCA = 2\ DCA + 2\ ACB - 2\ DCA\ (= 2\ ACB.)$

In each of these cases the angle AOB is measured by the arc AB, consequently angle $ACB = \frac{1}{2}\ AOB$ is measured by the half of the arc AB.

**173.** *Cor.* 1. All angles inscribed in the same or in equal segments are equal; because each is measured by the half of the same or of equal arcs; for example, angle $CED = CGD$, (fig. 106.)

**174.** *Cor.* 2. Every angle, as ACB, ADB, (fig. 103,) inscribed in a semi-circle is a right angle, for it has for its measure the half of a semi-circumference, that is, a quadrant.

**175.** *Cor.* 3. Every angle, as ACD, CDB, (fig. 103,) inscribed in a segment less than a semi-circle is

an obtuse angle, for it is measured by the half of an arc greater than a semi-circumference.

**176.** *Cor.* 4. Every angle, as ABC, ABD, inscribed in a segment greater than a semi-circle is an acute angle, for it is measured by the half of an arc less than a semi-circumference.

**177.** *Cor.* 5. In every quadrilateral inscribed in a circle, the sum of the opposite angles = 2 R. A. (Fig. 105,) $A + C = 2$ R. A.; $B + D = 2$ R. A. For the angle A is measured by $\frac{\text{arc DCB}}{2}$; the angle C is measured by $\frac{\text{arc DAB}}{2}$; therefore $A + C$ are measured by $\frac{\text{arc DCB} + \text{arc DAB}}{2}$ = a semi-circumference, which is the measure of 2 R. A.

**178.** *The angles made by a tangent and a chord, drawn from the point of contact, are equal to the angles inscribed in the alternate segments of the circle.* (fig. 106.) Angle DCB = CED and angle ACD = CHD; that is, each angle made by the tangent and chord is equal to the angle inscribed in the segment on the opposite side of the chord.

*Dem.* Draw the diameter CG. Angle GCB is a right angle (**171**;) therefore angle GCD + DCB = GCB = 1 R. A. Angle GDC is a right angle (**172**;) therefore angle GCD + DGC = 1 R. A. (**118**;) consequently angles GCD + DCB = GCD + DGC; therefore angle DCB = DGC. Angle DGC = DEC, therefore DCB = DEC. Again, angle CED + CHD = 2 R. A. (**177**;) angles ACD + DCB = 2 R. A., conse-

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quently angle CED + CHD = ACD + DCB; CED = DCB; consequently CHD = ACD.

**179.** *The radius of a circle may be drawn six times as a chord of the same circle.*

*Dem.* (Fig. 107.) In a circle draw the chord AB equal to a radius of the circle. Draw the radii OA and OB. We shall thus have an equilateral triangle OAB, consequently the angle AOB = $\frac{2}{3}$ R. A. = 60°. The angles about the centre O are together equal to 4 R. A. = 360°; consequently six angles each equal to AOB may be made about this centre. Equal angles at the centre intercept equal arcs; and the chords which subtend those arcs are equal (**166**;) consequently each of the chords of these 6 arcs is equal to AB.

**180.** *Cor. To inscribe a regular hexagon in a circle,* we have only to draw the radius six times as a chord. The angles of the hexagon will be equal; since each is measured by the half of an arc equal to $\frac{4}{6}$ of the circumference. Thus the hexagon having equal sides and equal angles is a regular hexagon.

**181.** The preceding proposition enables us to solve a problem of great practical utility, viz., to find very nearly how often the radius or diameter of a circle is contained in the circumference of the same circle; that is the *ratio* which one bears to the other. Since a straight line can never coincide with a curved line, this cannot be found directly; nor with perfect accuracy; but by comparing the circumference of a circle with the perimeter of an inscribed regular polygon, the ratio has been found by *approximation*, as it is called. It is first compared with an inscribed hexagon. A chord being a

straight line is less than the arc which it subtends; therefore the circumference is more than six times as great as a radius of the same circle, or more than 3 times as great as a diameter; that is, the diameter multiplied by 3 would not give us the length of the circumference. The smaller the chord the more nearly will it approach in magnitude to the arc it subtends; the greater the number of the sides of the inscribed polygon the more nearly will its perimeter approach to the circumference of the circumscribing circle; so that if the number of the sides were infinite the difference between the perimeter and circumference would be infinitely small. (Fig. 107.) From the centre O let fall a perpendicular upon the chord EF; thus the chord and the arc EF will both be bisected, at the points I and G respectively (**165.**) Draw GE and GF; we thus have 2 sides of an inscribed regular *dodecagon.* By means of the right-angled triangles OIF and GIF, by a process which will hereafter be explained (**220,**) we can find the magnitude of a side of this dodecagon. The side of the dodecagon is then bisected in like manner, and the magnitude of the side of a regular inscribed polygon of 24 sides is found. We thus *approximate* more and more nearly to the truth. The process has been continued until the perimeter of the inscribed polygon consisted of many thousand sides. The diameter of the circumscribed circle being taken equal to 1, the calculation of the perimeter of the inscribed polygon has been carried to 140 decimals; but the value 3.1415926 is near enough for all purposes; that is, the circumference of a circle is 3.1415926 times greater than its diameter. For ordinary purposes the expression 3.14 $=\frac{314}{100}$ will be found sufficiently accurate. If the cir-

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cumference is divided into 314 parts, the diameter will contain 100 of these parts; and by *multiplying the diameter of any circle by* $\frac{314}{100}$ *we obtain the circumference of that circle.* For example, let the diameter of a circle be 10 feet; the circumference is $\frac{10 \times 314}{100} = 31\frac{4}{10}$ feet. *If the circumference is known, the diameter may be found by multiplying the circumference by* 100 *and dividing the product by* 314. For example, if the circumference of a circle is 50 feet, the diameter is $\frac{50 \times 100}{314} = 15\frac{290}{314}$ feet.

### 8. Mensuration of Circles.

**182.** The *quadrature* or *squaring* of the circle, that is, the finding the area of a circle, is a problem upon which a great deal of time has been wasted. It never can be solved *exactly*. We can obtain the approximate area by considering the circle as a regular polygon having an infinite number of sides. The area of such a polygon is found by multiplying its perimeter by half the radius of the inscribed circle. But the perimeter of a polygon having an infinite number of sides may be considered as coinciding with the circumference of the inscribed circle, since the difference will be infinitely small. Hence the radius of the inscribed circle and the radius of that to be measured will be equal; and therefore *the area of a circle may be found by multiplying its circumference by half its radius.*

For example, if the diameter of a circle is 8 feet, what is the area of the circle? *Answer.* The circum-

ference of the circle will be $\frac{314 \times 8}{100} = 25\frac{3}{25}$ feet. Multiply $25\frac{3}{25}$ feet by $\frac{1}{2}$ the radius or $\frac{1}{4}$ the diameter, and we have $25\frac{3}{25} \times 2 = 50\frac{6}{25}$ square feet for the area of the circle.

**183.** The process for finding the area of the *sector* of a circle, that is, the portion contained between an arc and the radii drawn from its extremities as OADB, (fig. 99,) is easily deduced from the preceding proposition. The arc may be considered as made up of infinitely small straight lines, from the extremities of which radii may be drawn. The sector will thus be divided into triangles, and its area will be equal to the sum of the products of the bases of the triangles by $\frac{1}{2}$ their altitudes. The triangles have a common altitude equal to the radius, and the sum of their bases is equal to the arc of the sector; therefore *the area of the sector of a circle is equal to the product of its arc by half its radius.*

**184.** The area of the segment ABD, (fig. 99,) is found by subtracting the area of the triangle AOB from the area of the sector OADB.

**185.** The area of an annular surface is found by subtracting the area of the smaller circle from that of the greater.

**186.** It is often required to measure a figure bounded in part by straight and in part by curved lines, as ABCDEH (fig. 108.) Draw the straight line HD. The area of the pentagon ABCDH is equal to the sum of the areas of the triangles into which it may be divided by the diagonals HB and HC. In the curve

line DH take any points E, F, G, so that the parts DE, EF, &c., may without any material error be considered as straight lines. Draw HE and HF. Find the areas of the triangles HED, HFE, and HGF, and their sum added to the area of the pentagon will be the area of the entire figure.

## V. Solids.

### 1. Of Solids in General.

**187.** We have supposed a surface to be the space described by the motion of a line; in like manner a solid may be supposed to be produced by the motion of a surface in any direction but that of its length or breadth. The surface itself has extension in two directions, and the motion produces an extension in a third direction, viz., in thickness; thus a solid has three dimensions of extension, viz., length, breadth, and thickness. As lines are terminated by points, and surfaces by lines, so solids are terminated by surfaces. These surfaces may be either plane or curved. For example, a triangle, quadrilateral, or other rectilineal figure, moved in a direction perpendicular to itself, generates the prism, a solid bounded by plane surfaces. A right-angled triangle, revolving on one of those sides which include the right angle, generates a cone, a solid bounded by one curved and one plane surface. A rectangle revolving on one of its sides generates a cylinder, a solid bounded by one curved and two plane surfaces. A circle or a semi-circle revolving on its diameter generates the sphere, a solid bounded by one curved surface. The three solids last named, viz., the cone, the cylinder,

and the sphere, are called *the three solids of revolution*, or *the three round bodies.*

### 2. Mensuration of the Surfaces.

*The Prism and Cylinder.*

**188.** The convex surface of a prism is composed of parallelograms having an equal altitude. The bases of the prism are equal polygons. *Find the area of each face, and the sum of the whole will be the superficial contents of the prism.*

**189.** Since the *cube* is bounded by 6 squares, and consequently by equal faces, its superficial contents may be found by taking the area of one face 6 times.

For example, if the side of a cube $= 2$ feet, then the area of one face $= 2 \times 2 = 4$ square feet, and the superficial contents of the whole cube will be equal to $4 \times 6 = 24$ square feet.

**190.** The surface of a perpendicular *cylinder* consists of two equal circles, and of a convex surface which is equivalent to a parallelogram, whose base is equal to the circumference of one of the bases of the cylinder, and whose altitude is equal to the altitude of the cylinder, (fig. 6.) Therefore *the area of the convex surface of the cylinder is found by multiplying the circumference of one of the bases by the altitude of the cylinder. To this product add the areas of the bases, and we shall have the superficial contents of the whole cylinder.*

*The Pyramid and Cone.*

**191.** The base of a pyramid is a polygon, and its convex surface is composed of triangles. Find the area

of each separately, and the sum of the whole will be the superficial contents of the pyramid.

If the base is a regular figure, and the pyramid is perpendicular, that is, if a perpendicular let fall from the *vertex* (or point where all the triangles meet) to the base, passes through the centre of the base, then all the triangles of the convex surface will be equal, and its area may be found by multiplying the sum of the sides of the base by $\frac{1}{2}$ the altitude of one of the triangles.

**192.** The *Cone* may be considered as a perpendicular pyramid whose convex surface is composed of an infinite number of equal triangles. Consequently *the convex surface of the cone is equal to the product of the circumference of its base multiplied by half its side*, as the straight line AB, (fig. 11,) drawn from the vertex of the cone to the circumference of the base, is called.

By examining the skeleton of the cone, (fig. 11,) you will find that its convex surface is equal to a sector of a circle, whose radius is equal to the side of the cone, and whose arc is equal to the circumference of the base of the cone. The area of the sector is found by multiplying its arc by half its radius, consequently the area of the convex surface of the cone is found by multiplying the circumference of its base by half its side. To this product add the area of the base, and we have the superficial contents of the cone.

*Example.* What are the superficial contents of a cone whose side is 12 feet, and the radius of whose base is 5 feet? *Answer.* The circumference of the base is $\frac{314 \times 10}{100} = 31\frac{2}{5}$ feet; therefore the area of the base is

5/2 $31\frac{2}{5} \times \frac{2}{5} = 78\frac{1}{2}$ square feet. The convex surface is

$31\frac{2}{5} \times \frac{12}{2} = 188\frac{2}{5}$ square feet. The superficial content of the cone is $78\frac{1}{2} + 188\frac{2}{5} = 266\frac{9}{10}$ square feet.

### *Regular Solids.*

**193.** All the faces of the regular solids are regular, and therefore equal figures. Consequently the *superficial contents of a regular solid are equal to the area of one of its faces multiplied by the number of faces.*

### *The Sphere.*

**194.** As a regular polygon is inscribed in a circle, and the circumference of the circle compared with the perimeter of the polygon, in order to get the approximate ratio of the circumference to the diameter of a circle; so we can suppose a regular *polyedron*, as a solid bounded by plane faces is called, to be enclosed within the sphere, and the superficies of the sphere to be compared with that of the polyedron. In this manner the method of calculating *the superficial contents of the sphere* has been learnt. It is found that *it is equal to the product of the diameter of the generating circle multiplied by its circumference.* This circle, and indeed all circles, whose centres coincide with that of the sphere, are called *great* circles of the sphere. The *diameter* of a sphere is a line passing through the centre and terminating both ways in the surface.

The area of a circle is equal to the product of its circumference by half its radius, or one fourth of its diameter, (**182**;) consequently the surface of a sphere is equal to four times the area of a great circle.

*Example.* If the diameter of the sphere is 10 inches, then the circumference of a great circle of that sphere

will be $\frac{314 \cdot \times 10}{100} = 31\frac{4}{5}$ inches; and the area of a great circle will be $31\frac{4}{5} \times \frac{10}{4}$ square inches. The surface of the sphere will be $31\frac{4}{5} \times \frac{10}{4} \times 4 = 31\frac{4}{5} \times 10$ square inches; that is, the surface of the sphere is equal to the circumference of a great circle multiplied by the diameter of the sphere.

### 3. Mensuration of the Solidity.

**195.** The *Cube* is the most simple of solids, since its three dimensions are the same. Therefore, a cube whose side is a linear unit has been adopted as the *unit of solidity.* As its sides are linear units, its faces are the squares of those linear units, that is, they are superficial units. If the side be one inch, the face will be one square inch, and the whole body one cubic inch; if the side be a foot, the face will be a square foot, and the whole solid a cubic foot. To measure a solid we seek how many of these units may be contained in the solid, or into how many cubes of equal magnitude with that taken for the unit, the solid to be measured may be divided. As we used the word *area* to denote a surface considered as measured; in a like manner we use the word *solidity* to denote the magnitude of a solid or its bulk. The terms *volume* and *solid contents* have a similar import.

#### *The Prism and the Cylinder.*

**196.** If the parallelograms composing the convex surface of a prism are perpendicular to the base, it is called a *right* prism; otherwise it is called an *oblique*

prism. If the bases of a prism be parallelograms, then all its faces will be parallelograms, and the prism is called a *parallelopiped.* If all the faces are rectangles it is called a *rectangular parallelopiped.*

**197.** We will first seek the method of finding the solidity of a rectangular parallelopiped. Let us suppose its base to be 4 feet long and 3 feet broad, then the area of this base will be $4 \times 3 = 12$ square feet.

If the parallelopiped be 1 foot thick or high, it is apparent that it may be divided into 12 lesser solids, each of which will measure 1 cubic foot; consequently the whole parallelopiped will measure 12 cubic feet; which is equal to $4 \times 3 \times 1$, that is, to the product of its three dimensions. If the parallelopiped is 2 feet high, then there may be two layers of the small solids, that is, twice as many as before, and the solidity of the parallelopiped will be $4 \times 3 \times 2$, that is, the product of its three dimensions. Hence we infer that the solidity of a rectangular parallelopiped may be found by multiplying together its three dimensions; or, since the product of its length and breadth is the area of the base, it may be expressed shortly; *The solidity of a rectangular parallelopiped is equal to the product of its base by its altitude.*

The *altitude* of a prism is a perpendicular let fall from one base to the other, or to the other produced. In a right prism the altitude is equal to each of the upright sides. In this oblique prism (fig. 110) the line EG is the altitude.

**198.** The *cube* is a rectangular parallelopiped, of which the three dimensions are the same, consequently the solidity of a cube may be found by multiplying one side into itself twice. For example, if the side of a cube

is 8 inches long, then its solidity is $8 \times 8 \times 8 = 512$ cubic inches. Thus 1 cubic foot $= 12 \times 12 \times 12 =$ 1728 cubic inches.

**199.** Let us suppose a rectangular and an oblique parallelopiped of equal altitude, to be constructed upon the same base. The rectangular parallelopiped is formed of an infinite number of rectangles, the bases of which compose the base of the parallelopiped; and the oblique parallelopiped is formed of an infinite number of oblique parallelograms, the bases of which compose the base of the parallelopiped. These component rectangles and
151 parallelograms are equivalent, (~~110~~,) their sums must be equivalent, that is, the two parallelopipeds are equivalent. The solidity of a rectangular parallelopiped is found by multiplying its base by its altitude; consequently, *the solidity of any parallelopiped may be found by multiplying its base by its altitude.*

**200.** Every parallelopiped may be divided into two equal prisms, the bases of which are equal triangles. The solidity of each of these triangular prisms is one half the solidity of the whole parallelopiped; it may therefore be found by multiplying the altitude of the parallelopiped by one half its base. But each of the triangles which are the bases of the triangular prism is one half of the base of the parallelopiped; consequently, *the solidity of the triangular prism may be found by multiplying its base by its altitude.*

**201.** Every prism may be divided into as many triangular prisms, as the polygon taken for its base can, by diagonals, be divided into triangles. The solidity of the entire prism is equal to the sum of the products

of the base of each triangular prism multiplied by its altitude. But the triangular prisms and the entire prism have an equal altitude, and the sum of all the bases of the triangular prisms is equal to the base of the entire prism; consequently, *the solidity of any prism is equal to the product of its base multiplied by its altitude.*

**202.** The *cylinder* may be considered as a prism whose convex surface is composed of an infinite number of parallelograms; consequently *the solidity of the cylinder may be found by multiplying its base by its altitude.* By the *altitude* is to be understood a perpendicular which measures the distance between the two bases.

*Example.* If the altitude of a cylinder is 8 inches, and the diameter of its base 6 inches, what is the solidity of the cylinder? *Answer.* The circumference of the base is $\frac{314 \times 6}{100} = 18\frac{84}{100}$ inches; and the area of the base $18\frac{84}{100} \times \frac{6}{4} = \frac{471 \times 6}{100} = 28\frac{13}{50}$ square inches. Consequently the solidity of the cylinder is $28\frac{13}{50} \times 8 = 226\frac{2}{25}$ cubic inches.

### *The Pyramid and the Cone.*

**203.** Every triangular prism may be divided into 3 triangular pyramids. Experiment will best show how this may be done. In this figure (fig. 110) we have a representation of it. ABC and DEF are the bases of a triangular prism, which is divided by the planes ABF and DFB into 3 triangular pyramids. These pyramids are *equivalent*, that is, equal in solidity, one to the other. It may be demonstrated that pyramids which have equal bases and equal altitudes are equivalent. It is

evident that the bases and altitudes of the two pyramids F-ABC and B-DEF are equal, consequently these two pyramids are equivalent. Let us now compare the pyramid F-ADB with B-DEF, of which we now suppose DEB to be the base. It is at once evident that these two pyramids have equal bases and equal altitudes, and therefore are equivalent, consequently F-ADB = B-DEF = F-ABC.

The triangular pyramid is, therefore, $\frac{1}{3}$ of a triangular prism of equal base and altitude. The solidity of the triangular prism is equal to the product of its base by its altitude, consequently, the solidity of a triangular pyramid is equal to $\frac{1}{3}$ of the product of its base by its altitude, *or*, to the product of its base by $\frac{1}{3}$ of its altitude.

**204.** Every pyramid may be divided into triangular pyramids; consequently, *the solidity of any pyramid is equal to the product of its base multiplied by $\frac{1}{3}$ of its altitude.*

**205.** The *Cone* may be considered as a pyramid whose convex surface is composed of an infinite number of triangles, consequently, *the solidity of a cone is equal to the product of its base multiplied by $\frac{1}{3}$ of its altitude.*

*Other Solid Bodies.*

**206.** The *Tetraedron* is a triangular pyramid. The *Octaedron* may be divided into 8 pyramids having equal bases and altitudes. Each of these pyramids will have a face of the polyedron for its base, and $\frac{1}{2}$ of a face axis for its altitude. The vertices lie together at the middle of the octaedron. The solidity of the octaedron is equal to 8 times the solidity of each pyramid. In a similar manner the dodecaedron may be divided into

12, and the icosaedron into 20 pyramids. In general *the solidity of any polyedron may be found by dividing it into pyramids; find the solidity of each pyramid, and the sum of all will be the solidity of the entire solid.*

*The Sphere.*

**207.** Let us suppose the sphere to be converted into a polyedron with an infinite number of faces. We can suppose each of these faces to be the base of a pyramid, with its vertex at the centre of the sphere. Each of these pyramids will have a radius of the sphere for its altitude. Thus the solidity of the sphere is equal to that of a pyramid having a base equivalent to the surface of the sphere, and an altitude equal to the radius of the sphere. Consequently the solidity of the sphere may be found by multiplying its surface by $\frac{1}{3}$ its radius or $\frac{1}{6}$ its diameter. The surface of a sphere is 4 times the area of a great circle of the same sphere; therefore the solidity of a sphere is equal to the product of a great circle of that sphere multiplied by $\frac{2}{3}$ the diameter.

*Example.* If the diameter of a sphere is 4 feet, what is its solidity? *Answer.* The circumference of a great circle is $\frac{314 \times 4}{100}$ feet; therefore the area is $\frac{314}{100} \times 4$ *square* feet $= 12\frac{14}{25}$ square feet; and the solidity is $12\frac{14}{25} \times 2\frac{2}{3}$ $= 33\frac{37}{75}$ cubic feet.

**208.** The surface of a sphere is equal to the product of the circumference of a great circle multiplied by the diameter; and the product of this surface multiplied by $\frac{1}{6}$ the diameter is the solidity of the sphere. Thus the solidity of a sphere may be found by multiplying the circumference of a great circle by $\frac{1}{6}$ the square of the

diameter. Again, the circumference of a great circle is equal to the product of the diameter multiplied by $\frac{314}{100}$; consequently the solidity of the sphere is equal to the product of $\frac{1}{6}$ of the cube of the diameter multiplied by $\frac{314}{100}$, which is equal to the cube of the diameter multiplied by $\frac{314}{600}$, or by its equivalent decimal fraction, .523. That is, the solidity of a sphere is equal to the product of the cube of its diameter multiplied by .523.

### Miscellaneous Propositions.

**209.** *To find the distance between two objects, when it cannot be directly measured,* as, for example, between A and B, (fig. 111,) which are separated by a sea.

*Sol.* Take any point C, so that the lines CA and CB can be directly measured with a rod or a chain. Produce these lines beyond C, taking CD = CA, and CE = CB. Because CA = CD; CB = CE and angle ACB = DCE, (**107**;) therefore triangle ACB = DCE, (**126**;) consequently AB = DE. Measure DE and we have the required distance AB.

**210.** *To find the distance between two objects, only one of which can be reached.*

Suppose the line AB, (fig. 111,) the point A being accessible.

*Sol.* Draw from A a straight line in the direction of AB. Draw AC perpendicular to such line, and of any convenient length; produce AC, taking CD = AC; upon CD at the point D erect a perpendicular; through C

draw a straight line in the direction BC and produce it until it intersects at E the perpendicular erected upon CD. Now AC = CD, angle A = D (being R. A.,) and angle ACB = DCE, (**107**;) therefore triangle ACB = DCE, (**127**;) consequently AB = DE. The length of AB is found by measuring DE.

**211.** *To divide a given triangle* ABC, (fig. 112,) *into two equivalent parts by a straight line.*

*Sol.* 1. Draw the line AD from the vertex of one of the angles A to the middle of the opposite side. Now BD = DC; that is, the triangles ABD and ADC have equal bases, and since their altitudes are also equal, the triangles are equivalent, (**153.**)

*Sol.* 2. Suppose it is required to draw the dividing line from the point D in one of the sides AB of the given triangle. Bisect the side AB, (fig. 113,) at E. Draw CE and CD. Now the triangle BDC is greater than ½ ABC by the part EDC; therefore we must take from triangle BDC a part equivalent to EDC. Draw EG parallel to DC; and then draw DG, which will be the division line required. For triangle EDC = CDG; since they have the same base CD, and an equal altitude. If therefore we take the triangle CDG from the triangle CDB, we shall take a part equivalent to EDC. There will remain the triangle BDG = BEC; and consequently = ½ ABC. Thus the line DG divides the triangle ABC into two equivalent parts.

**212.** *To transform a parallelogram,* ABCD, (fig. 114,) *into a triangle of equal area.*

*Sol.* Produce one side of the parallelogram BC, taking CE = BC. Draw AE. The triangle ABE is the one required.

*Dem.* Draw AC. Triangle ABC = ACE and triangle ABC = ACD, (**153**;) therefore triangle ACE = ACD. Add triangle ABC to each side of this equation, and we have ACE + ABC (= ABE) = ACD + ABC (= parallelogram ABCD.) That is, *if a triangle has an equal altitude and twice as great a base as a parallelogram, the triangle will be equivalent to the parallelogram.*

**213.** *To construct a triangle which shall be* $1\frac{1}{2}$ *times as great as a given parallelogram* ABCD.

*Sol.* (Fig. 115.) Produce BC and take BE = 3 BC. Draw BD and DE. BDE is the required triangle.

*Dem.* Draw DF. Triangle BCD = DCF = DFE, (**153.**) Triangle BCD = $\frac{1}{2}$ parallelogram ABCD; therefore BDE = $3 \times \frac{1}{2}$ ABCD = $1\frac{1}{2}$ ABCD.

**214.** *To transform a triangle into a parallelogram of equal area.*

*Sol.* Construct a parallelogram upon $\frac{1}{2}$ the base of the triangle, with an altitude equal to that of the triangle. It can readily be demonstrated that such a parallelogram will be equivalent to the triangle.

If it is required that the parallelogram should have an angle of a given size; then bisect BC, (fig. 116,) at D. Make at D, in line DC, an angle equal to the given angle. Through A draw AF parallel to BC; and through C draw CF parallel to DE. DCEF is the parallelogram required.

**215.** *In a parallelogram two intersecting diagonals bisect each other.*

*Dem.* (Fig. 117.) AB = DC, (**147**;) angle EAB

= ECD and angle EBA = EDC, (**110,**) consequently triangle AEB = DEC; therefore EB = ED and EA = EC.

**216.** *Two diagonals drawn in a square or a lozenge cut each other at right angles; in all other parallelograms at oblique angles.*

*Dem.* Let ABCD, (fig. 118,) be a square or a lozenge; then AE = EC, (**215,**) DE = DE; AD = DC by supposition; consequently triangle AED = CED, ([redacted];) therefore angle AED = CED; and conse- 138
quently DE is perpendicular to AC (**2.**)

2. Let ABCD, (fig. 117,) be a rectangle or a rhomboid, in which the sides AB and DC are greater than AD and BC. AE = AE, ED = EB, (**215,**) but AB > AD; therefore angle AEB > AED, (**144.**) But angle AEB + AED = 2 R. A. (**106**;) therefore AEB > 1 R. A., and AED < 1 R. A., that is, AC cuts DB obliquely.

**217.** *Two diagonals drawn in a square or a lozenge divide the angles into equal parts; in all other parallelograms into unequal parts.*

*Dem.* Let ABCD, (fig. 118,) be a square or a lozenge; then AB = AD; AC = AC; BC = DC; therefore triangle ABC = ADC; consequently angle CAB = CAD.

In parallelogram ABCD, (fig. 117,) side AB > BC; therefore angle ACB > CAB, ([redacted]) But angle ACB 139
= CAD, (**110**;) therefore angle CAD > CAB.

**218.** *To divide a parallelogram into 2, 3, 4, &c. equal parts.*

*Sol.* (Fig. 119.) Divide any side AB into the required number of parts, and through the division points

draw lines parallel to the side AD; the given parallelogram will thus be divided as was required.

**219.** *Let* GF (fig. 120) *be a straight line drawn through the middle of a diagonal of a parallelogram; how will it divide the parallelogram?*

*Sol.* Triangle ADC = ABC, (**146.**) EC = EA, angle ECG = EAF and CEG = AEF, (**107**,) therefore triangle EGC = EFA. Consequently triangle ADC — EGC (= quadrilateral AEGD) = triangle ABC — EFA (= quadrilateral ECBF.) Again, triangle EGC = EFA, therefore AEGD + EFA = AFGD = ECBF + EGC = BFGC; that is, *the parallelogram* ABCD *is divided into two equal parts.*

**220.** Let ABC (fig. 121) be a right triangle of which B is the right angle. Upon each side of the triangle construct a square. *What ratio will the square constructed upon the hypothenuse bear to the squares constructed upon the sides which include the right angle?*

*Sol.* If the side AB = BC, the squares constructed upon those sides will be equal, and a perpendicular let fall from the vertex of the angle B upon AC will bisect AC, (**133**,) and if produced will bisect the square ACDE. If AB > BC, then square ABFG > BCKH, and the perpendicular let fall from the vertex of the angle B will divide AC and also the square ACDE into 2 unequal parts; the greater part being nearest to A.

Let fall upon AC the perpendicular BL, and produce it to M. Draw BD and AK. The triangle ACK = ½ BCKH, (**152.**) Triangle BCD = ½ CLMD. In triangles ACK and BCD we have AC = CD and CK = CB; angle BCK = ACD being right angles and BCA = BCA; therefore angle BCK + BCA (= angle ACK)

= ACD + BCA (= BCD;) therefore triangle ACK = BCD, (**126**,) consequently twice the triangle ACK (= BCKH) is equal to twice the triangle BCD (= CLMD.) It may be demonstrated in a similar manner, that the parallelogram ALME = square ABFG. Consequently CDML + LMEA = square ACED = square AFGB + BCKH. That is, *in every right-angled triangle the square of the hypothenuse is equal to the sum of the squares of the other two sides.* This is called the proposition of Pythagoras, because he first discovered the ratio.

**221.** *Cor.* 1. The square of the hypothenuse of a right-angled isosceles triangle is equivalent to twice the square of one of the equal sides.

*Cor.* 2. *A square circumscribed about a circle is double a square inscribed in the same circle.* For the side of the former is equal to a diagonal of the latter, and is consequently the hypothenuse of a right-angled isosceles triangle.

**222.** *To construct a square which shall be equal to 2 given squares.*

*Sol.* (Fig. 122.) Let M and N be the sides of the given squares. Construct a right angle; take one of its sides equal to M, and the other side equal to N. Connect the extremities of these sides by the line X. The square constructed upon X will equal the sum of the squares of the sides M and N.

**223.** If it is required *to construct a square which shall be equal to the sum of* 3, 4, 5 *or more squares*, a similar method is adopted. Suppose, for example, the sides of the given squares are M, N, O, P, and Q, (fig.

122.) Make a right angle. Take one of its sides equal to M, the other equal to N. Draw X. Upon X, at its extremity, erect a perpendicular, and take in it a part equal to O. Draw Y. Upon Y, at its extremity, erect a perpendicular equal to P. Draw Z. Erect a perpendicular at the extremity of Z. Draw V. Now $\overline{V}^2 = \overline{Q}^2 + \overline{Z}^2$; $\overline{Z}^2 = \overline{P}^2 + \overline{Y}^2$; $\overline{Y}^2 = \overline{O}^2 + \overline{X}^2$; $\overline{X}^2 = \overline{M}^2 + \overline{N}^2$; consequently $\overline{V}^2 = \overline{M}^2 + \overline{N}^2 + \overline{O}^2 + \overline{P}^2 + \overline{Q}^2$.

**224.** *To construct a square which shall be* 4, 9, 16, 90, 100, 105, *&c., times as great as a given square.*

*Sol.* Produce the sides of the given square, until the produced parts shall be 2, 3, 4, 10, &c., times as great as the sides themselves; upon these parts construct squares, and such squares will be quadruple, nine times &c., the given square.

If it is required to construct a square which shall be 90 times as great as a given square, the operation will be somewhat more intricate. The shortest method is to take the next smallest square of a whole number, which is 81. Now $90 = 81 + 9$. Construct squares 81 times and 9 times as great as the given square, and place the two squares together at right angles with each other. Draw a line connecting the sides of the two squares which include the right angle, and such line will be a side of the required square. For example, (fig. 123,) $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 = 9 \times 1 + 81 \times 1 = 90 \times 1$; the sides of the given square being equal to 1 part, a square constructed upon AC will be the square required. Suppose it is required to construct a square 105 times as great as a given square. $105 = 100 + 4 + 1$. Make a right angle; take one of the sides 10 parts long; and the other side 2 parts long; draw the hypothenuse. At the

extremity of this erect a perpendicular 1 part long. Draw again an hypothenuse, and it will be a side of the required square.

**225.** *To construct a square which shall be equal to the difference of two given squares.*

*Sol.* (Fig. 122.) Let N and X be sides of the given squares. Make a right angle. Take one side of it equal to N, a side of the lesser square. From the extremity of this side of the angle as a centre, with a radius equal to X, a side of the greater square, describe an arc, which will intersect the other side of the angle. Thus we get M, which is a side of the required square. For $\overline{M}^2 + \overline{N}^2 = \overline{X}^2$; therefore $\overline{M}^2 = \overline{X}^2 - \overline{N}^2$.

**226.** *The angle formed by the intersection of two chords* (produced) *without a circle is equal to the difference between the two angles in the circumference standing upon the arcs intercepted by the two chords.*

*Dem.* Draw BC. (Fig. 124.) Angle ABC$=$BEC $+$BCE, (**124**,) therefore angle BEC$=$ABC$-$BCE.

**227.** *An angle formed by the intersection of two chords,* (AB and CD,) *within a circle is equal to the sum of the angles in the circumference standing upon the arcs intercepted by the chords.*

*Dem.* (Fig. 125.) Draw the chords AC and BD. Angle AED$=$ACD$+$CAB, (**124.**) In like manner angle CEB$=$ABD$+$CDB.

**228.** *To draw a tangent to a circle, whose centre is* O, *from a point* D *without the circle,* (fig. 126.)

*Sol.* Draw DO. Upon DO as a diameter describe a

circumference, which will intersect the circumference of the given circle at two points, A and B. Draw DA and DB, both of which lines are tangents to the given circle.

To demonstrate this let us draw the radii OA and OB. Now each of the angles OAD and OBD is a right angle, (**174.**) OA is perpendicular to AD and OB to BD; and consequently AD and BD are tangents, (**170.**)

**229.** Again, OD = OD; OA = OB; angles OAD and OBD are right angles; therefore triangle OAD = OBD, (**135**;) consequently AD = BD; that is, *two tangents to a circle drawn from the same point, and not produced beyond the point of contact, are equal.*

**230.** Since the triangles OAD and OBD are equal, angle ODA = ODB; that is, *the angle formed by the intersection of two tangents to the same circle is bisected by a straight line drawn from the centre of that circle to the vertex of the angle.*

**231.** *To transform a rectangle into a square.*

*Sol.* Upon the greatest side AB of the rectangle ABCD, (fig. 127,) construct a square ABFG. Upon AF as a diameter describe a semi-circle. Produce CD to the circumference at E. Draw AE. A square constructed upon AE will be equivalent to the rectangle ABCD.

*Dem.* Draw EF. Angle AEF is a right angle, (**174**,) therefore $\overline{AE}^2 + \overline{EF}^2 = \overline{AF}^2 = ABGF$. ED is perpendicular to AF, therefore $\overline{EA}^2 = ABCD$, (**220.**)

**232.** *To draw a meridian line;* that is, to draw a horizontal straight line in the direction North and South. At noon the shadows of objects fall in the direction of their meridian line. Such a line is used in the construction of sun dials.

*Sol.* Erect upon an exactly horizontal plane a perpendicular rod. From the foot of the rod, as a centre, with radii of any length, describe two or more concentric circles. The length of the shadow of the rod will become less as the sun becomes higher; therefore there will be a moment when the end point of the shadow will lie in the circumference of the greater circle. Mark this point in the circumference. In the afternoon the length of the shadow will again increase, and there will be a moment when the end of the shadow will again lie in the circumference of the greater circle. Mark this point. Connect by a straight line the two points thus found. Bisect this straight line, and at the middle point erect a perpendicular. Such perpendicular will mark the direction in which the shadows fall at noon; it is therefore the meridian line. By proceeding in the same manner with the second circle the accuracy of the first observation will be tested.

**233.** *Upon a given straight line to describe a segment which shall contain a given angle;* that is, *a segment such that the angle having its vertex in the arc, and its sides standing upon the extremities of the segment, shall be equal to the given angle.*

Let AB be the given straight line and O the given angle, (fig. 129.)

*Sol.* At the point A in line AB make an angle BAC equal to the given angle O. Upon AC at A erect a perpendicular. Bisect AB and at the middle point G

erect a perpendicular. The two perpendiculars will intersect each other at D. From D as a centre with radius DA describe a circle: AEB is the required segment. Make any angle in the circumference of this segment, for instance, AEB; then is angle AEB = angle BAC = angle O.

*Dem.* Angle DAC = 1 R. A.; therefore angle DAG is < 1 R. A. Angle DGA = 1 R. A.; therefore angle DGA + angle DAG < 2 R. A. Consequently the perpendiculars intersect each other, (**112.**)

GA = GB, GD = GD, angle DGA = angle DGB;
136 therefore triangle DGA = triangle DGB, therefore DA = DB. Thus the circle which is described from D, with radius DA, passes also through B. Now AC is a tangent to the circle, (**170**;) therefore angle AEB = angle CAB, (**178**,) which was to be proved.

**234.** *To construct a triangle of which the base, the altitude, and the angle opposite to the base are given.*

(Fig. 128.) Let M be the base, N the altitude, and O the angle opposite to the base.

*Sol.* Draw a straight line; take AB = M; describe upon AB a segment capable of containing the angle O. Upon AB at any point G erect a perpendicular, and in it take GC = N. Through C draw a line parallel to AB. From the points D and E where this parallel cuts the arc of the segment draw the straight lines DA, DB, EA, EB; either of the triangles DAB and EAB is the triangle required.

**235.** *In every circle equal chords are at an equal distance from the centre; and if two chords are at an equal distance from the centre they are equal.*

*Dem.* (Fig. 130.) 1. Let AB = CD. Draw the

radii OA and OC. Because AB = CD, therefore ½ AB = AE = ½ CD = CF; OA = OC; therefore the right-angled triangles OEA and OFC are equal, (**135**;) therefore OE = OF.

2. Let OE = OF. In the right-angled triangles OEA and OFC we have OA = OC, and OE = OF; therefore triangle OEA = OFC; therefore AE = CF. But AE = ½ AB, and CF = ½ CD; therefore AB = CD.

**236.** *Of two unequal chords the greater is at the least distance from the centre.*

*Dem.* (Fig. 130.) Let HI > CD; therefore GI > CF. Now $\overline{GI}^2 + \overline{GO}^2 = \overline{OI}^2$, and $\overline{CF}^2 + \overline{OF}^2 = \overline{OC}^2$, (**220.**) Now $\overline{OI}^2 = \overline{OC}^2$; therefore $\overline{GI}^2 + \overline{GO}^2 = \overline{CF}^2 + \overline{OF}^2$. But $\overline{GI}^2 > \overline{CF}^2$; therefore $\overline{OG}^2 < \overline{OF}^2$; therefore OG < OF.

**237.** *Cor.* The diameter is the greatest chord in a circle, because it is at no distance from the centre.

**238.** *To describe a circle whose circumference shall pass through two given points, as* A *and* B, (fig. 129.)

*Sol.* Draw the straight line AB; this line will be a chord of the required circle. Bisect AB, and at the middle point G erect a perpendicular. A circle described from any point of this perpendicular with a radius equal to the distance between such point and A or B, will pass through the points A and B.

**239.** *To describe a circle whose circumference shall pass through three given points not in the same straight line, as* A, B, *and* C, (fig. 131.)

*Sol.* Draw the straight lines AB and BC, which will

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be chords of the required circle. At the middle of each of these lines erect a perpendicular. Both of these perpendiculars will pass through the centre of the required circle, (**169**,) which must therefore be the point O at which they meet. A circle described from this point with a radius equal to OA or OB, will be the one required.

**240.** We have here solved another problem, namely, to describe a circle about a given triangle.

**241.** *To describe a circle about a regular polygon* ABCDE, (fig. 97.)

*Sol.* The sides of the given polygon will be chords of the required circle. Find the centre of this circle by erecting perpendiculars upon the middle points of these chords. This centre being in each perpendicular is at the same distance from the extremities of each chord. This distance is therefore the radius of the required circle.

**242.** *To describe a circle whose circumference shall touch both the sides of a given angle.*

*Sol.* Let A be the angle, (fig. 132.) In the two sides take AC = AD, and at the points D and C erect perpendiculars which will intersect each other at O. From O as a centre, with a radius OC, describe a circle, which will be the circle required.

*Dem.* Draw OA. By construction AC = AD; AO = AO; angle ACO = ADO being right angles; therefore triangle ACO = ADO, therefore OC = OD. Therefore a circle described from O with a radius OC will pass through D and C; and since the radii OC and OD

are perpendicular to the sides of the angle, these sides will be tangents to the circle, (**170.**)

**243.** *Remark.* Since triangle ADO = ACO, therefore angle DAO = CAO; that is, an angle is bisected by a line drawn from its vertex to the point of intersection of perpendiculars erected upon its sides at equal distances from its vertex.

**244.** *To inscribe a circle in a triangle.*

*Sol.* (Fig. 133.) Bisect the angles A and B by straight lines, which will intersect at O. From O let fall upon the three sides of the triangle, the perpendiculars OD, OF, and OE. A circle described from O as a centre, with a radius OD, OF, or OE will be the circle required.

*Dem.* Angle ADO = AFO; angle DAO = FAO; AO = AO; therefore triangle DAO = FAO; therefore OD = OF.

Again, angle ODB = OEB; angle OBD = OBE; therefore DOB = BOE, (**123**;) OB = OB; therefore triangle OBD = OBE; consequently OD = OE. Thus OD = OE = OF, and a circle described from O, with a radius OD, will pass through the points D, E, and F. The radii are perpendicular to the sides of the triangle at these points, therefore the sides are tangents to the circle, (**170.**)

**245.** *To describe a circle whose circumference shall touch the circumference of another circle at a given point* A, *and shall also pass through a given point* C, (fig. 134.)

*Sol.* Draw OA, and produce it indefinitely. Draw AC and bisect it at B. At B erect a perpendicular

which will intersect OA produced at D. The point D is the centre of the required circle, and DA is a radius.

*Dem.* By construction BA = BC; BD = BD; angle ABD = CBD; therefore triangle ABD = CBD; therefore AD = CD; therefore a circle described from D as a centre, with radius DA, will pass through points A and C, and is therefore the circle required.

## VII. Of Proportions.

### 1. Of Geometrical Proportions in General.

**246.** The term *ratio* is used to denote the comparative magnitude of two quantities; for example, if one line is 2 inches long, and another line is 12 inches long, we say the lines are to one another in the ratio of 2 to 12, or of 1 to 6; that is, one is six times as great as the other. Two equal ratios form a *proportion.* If any quantity is contained in a second quantity, as often as a third is contained in a fourth, these four quantities form a *geometrical proportion.* For example, (fig. 142;) in the triangles ABC and DEF, let DE = 2 AB, and DF = 2 AC, and EF = 2 BC, we should have these proportions;

$$DE : AB = EF : BC$$
$$EF : BC = DF : AC$$
$$DE : AB = DF : AC$$

which we should read thus;

DE is to AB as EF is to BC, &c.,

meaning that the quotient of DE divided by AB is equal to the quotient of EF divided by BC. The first term of each ratio is called the *antecedent,* the second term is the *consequent.* The first and fourth terms of a propor-

tion are called the *extremes;* the second and third terms are the *means.* If the same quantity is taken twice as a mean it is called a *mean proportional* between the extremes.

*A proportion is the equality of ratios;* hence two or more ratios are required to form a proportion. Proportion does not require the equality of the quantities, but the equality of the ratios; hence two lines may form a proportion with two surfaces; for one line may be contained as often in the other line, as one surface in the other surface.

**247.** As a ratio expresses the number of times that one quantity is contained in another, it may be represented in the form of a fraction. Let us substitute numbers for lines, $\frac{4}{6} = \frac{8}{12}$. Now let us reduce these fractions to a common denominator and we shall have $\frac{4 \times 12}{6 \times 12} = \frac{8 \times 6}{12 \times 6}$. Now omit the common denominator, and we have $4 \times 12 = 8 \times 6$; that is, *the product of the extremes is equal to the product of the means.* This proposition is true of all proportions, and may be used as a *test* to ascertain if a proportion exists.

**248.** If all the terms are of the same kind, for example, all lines, or all numbers, we may transpose the means, and yet keep the proportion; thus, $4:6=8:12$ gives the new proportion $4:8=6:12$, by *transposition,* as it is called. Or, we may get a new proportion by *inversion,* that is, by changing the places of the terms of each ratio; thus, $6:4=12:8$. There are other changes which may be made without destroying the proportions, and they will be mentioned as occasion requires their use.

## 2. Of Similarity of Figures.

**249.** Figures are *similar* which have the same form, whether they are equal or unequal in magnitude. Figures may be similar in form, and yet be very different in magnitude; or they may be equal in magnitude, and yet be very different in form.

*Figures are similar when their corresponding angles are equal, and a proportion exists between their homologous sides.* By *homologous* sides are meant those which have the same position in the different figures with respect to the equal angles. The angles which are equal in the different figures are also called *homologous* angles.

Neither the equality of the angles, nor the proportion of the sides, will alone constitute similarity of figures, but the combination of the two, (fig. 135.) If the figures ABCD and EFGH are similar, then angle A = E, angle B = F, angle C = G, angle D = H, and these proportions exist:

$$AB : EF = BC : FG$$
$$BC : FG = CD : GH$$
$$CD : GH = DA : HE$$
$$DA : HE = AB : EF$$

Also, by transposition,

$$AB : BC = EF : FG$$
$$BC : CD = FG : GH$$
$$CD : DA = GH : HE$$
$$DA : AB = HE : EF$$

In these figures angles A and E, B and F, C and G, D and H have the same position; and the sides AB and EF, BC and FG, CD and GH, DA and HE have the same position with respect to these angles.

The figures would not be similar if other angles, not

homologous, were equal, and other sides, not homologous, were proportional. If, for instance, angle A = angle E, angle B = angle G; or if the proportion were AB : EF = BC : GH.

### 3. Triangles.

**250.** *What ratio do the triangles* ABC *and* DCE, (fig. 136,) *which lie between the same parallels*, that is, *have an equal altitude, bear one to another?* or in other words, *How often is the surface of triangle* ABC *contained in the surface of triangle* DCE? Or, *What two other quantities are contained as often one in the other as triangle* ABC *is contained in triangle* DCE?

*Answer.* Apply the base BC to the base CE as often as it can be applied, suppose $2\frac{1}{2}$ times, and join the division points F and G with the vertex D; then is triangle ABC = triangle DCF = triangle DFG = 2 × triangle DGE. Triangle ABC is contained $2\frac{1}{2}$ times in triangle DCE; that is, just as often as the base BC is contained in the base CE. Thus we have triangle ABC : triangle DCE = BC : CE; that is, *Triangles of equal altitude are to each other in the ratio of their bases.* If 2 triangles have equal altitudes, and the base of the one is 2, 3, 4, 10, $\frac{1}{2}$, $\frac{3}{8}$ times as large as the other, then is the surface of the one triangle contained 2, 3, 4, 10, $\frac{1}{2}$, $\frac{3}{8}$ times in the surface of the other.

**251.** Triangles are halves of parallelograms having an equal base and an equal altitude, (**152**,) and since the ratio of the halves and of the whole quantities must be the same, therefore *parallelograms of equal altitude are to one another as their bases.*

**252.** (Fig. 137.) *If the triangles* ABC *and* DBC *have equal bases, or a common base* BC, *what is the ratio of triangle* ABC *to triangle* DBC?

*Answer.* Erect upon BC, at the point B, the perpendicular BF. Through the points A and D draw lines parallel to BC and intersecting BF at the points E and F; then BE is the altitude of the triangle ABC, and BF is the altitude of triangle DBC. Draw EC and FC. Triangle ABC is equivalent to EBC, and triangle DBC is equivalent to FBC, (**153.**) We may therefore compare together triangles EBC and FBC, instead of the triangles ABC and DBC.

Now the triangles EBC and FBC have a common altitude BC; they are to one another therefore as their bases BE and BF; thus we have the proportion,

$$\text{EBC} : \text{FBC} = \text{EB} : \text{BF}.$$

*It is a principle of geometrical proportions that equal quantities may be substituted one for the other.* We have therefore the proportion,

$$\text{Triangle ABC} : \text{triangle DBC} = \text{EB} : \text{BF};$$

that is, *Triangles having equal bases are to each other as their altitudes.*

**253.** *Cor. Parallelograms having equal bases are to each other as their altitudes.*

**254.** *What ratio do the triangles* ABC *and* DEF (fig. 138) *bear to one another, the base and altitude of the one being respectively unequal to the base and altitude of the other?*

*Answer.* From the vertex of the angle A let fall upon the opposite side BC the perpendicular AG; and from the vertex of the angle D let fall upon the opposite side EF the perpendicular DH.

Construct a third triangle with a base MN = BC, and an altitude NO = DH.

Let us compare together the triangles OMN and ABC, which have equal bases, and the triangles OMN and DEF, which have equal altitudes. We thus find the ratio which two quantities each bear to a third quantity, and this being known, we can find the ratio which they bear to each other.

Triangle ABC : triangle MNO = AG : ON.

Triangle MNO : triangle DEF = MN : EF.

*Two proportions may be multiplied together, term by term, and the products will form a new proportion.*

Therefore triangle ABC $\times$ triangle MNO : triangle MNO $\times$ triangle DEF = AG $\times$ MN : ON $\times$ EF.

*If both the terms of a ratio be divided by the same number, the ratio remains unchanged.*

Therefore ABC : DEF = AG $\times$ BC : DH $\times$ EF, that is, *Triangles having unequal bases and unequal altitudes, are to one another as the products of their bases multiplied by their altitudes.*

**255.** From the preceding propositions may be deduced the general proposition, *that triangles are to one another as the products of their bases multiplied by their altitudes.*

This proposition may be deduced immediately from the method of calculating the area of a triangle, (**161.**)

Triangle ABC $= \frac{BC \times AG}{2}$; triangle DEF $= \frac{EF \times DH}{2}$;

therefore,

triangle ABC : triangle DEF $= \frac{BC \times AG}{2} : \frac{EF \times DH}{2}$;

Or, ABC : DEF = BC $\times$ AG : EF $\times$ DH.

**256.** If the areas of the triangles ABC and DEF are equal, then BC $\times$ AG $=$ EF $\times$ DH. We have here two equal products; and the factors of such products, being transposed, form a proportion.

Thus, BC : EF $=$ DH : AG.

That is, *If two triangles have equal areas, their bases are to one another in the inverse ratio of their altitudes, and the altitudes in the inverse ratio of the bases.* For example, if in two triangles having equal areas, the base of one is double that of the other, then the altitude of the last must be double the altitude of the first.

**257.** Since triangles are the halves of parallelograms of equal bases and equal altitudes, and halves are to one another as their wholes; therefore, *parallelograms are to one another as the products of their bases multiplied by their altitudes;* and *in parallelograms having equal areas the bases are to one another in the inverse ratio of the altitudes; and the altitudes are to one another in the inverse ratio of the bases.*

*Example.* If the base of a parallelogram is 12 feet, and its altitude 8 feet, what is the altitude of an equivalent parallelogram, whose base is 20 feet?

*Answer.* $20 : 12 = 8 : \frac{12 \times 8}{20} = 4\frac{4}{5}$ feet.

**258.** *Draw a straight line* DE, *intersecting two sides of the triangle* ABC, *and parallel to the third side* BC; *what ratio do the parts into which the sides are divided bear to one another, and to the entire sides?* (Fig. 139.)

*Sol.* Draw DC and EB. The triangles DEC and EDB have a common base and an equal altitude, they are therefore equal in area. For the same reason tri-

angle EBC = DBC. Equals added to equals give equals; therefore DEC + ADE = ADC = EDB + ADE = AEB.

Equal quantities bear an equal ratio to equal or the same third quantities; or, a third quantity bears the same ratio to two equal quantities.

Thus ADE : DEC = ADE : EDB
But ADE : DEC = AE : EC (**250**)
And ADE : EDB = AD : DB (**250**)

Therefore AE : EC = AD : DB; that is, *the parts of one side are proportional to the parts of the other side, directly.*

Again, triangle ADC = AEB, therefore

ADC : ADE = AEB : ADE.

From this we may deduce the proportion,

AC : AE = AB : AD.

And, because triangle ADC = AEB and triangle DEC = EDB, therefore

ADC : DEC = AEB : EDB;

from which we may deduce the proportion,

AC : EC = AB : DB.

That is, *Each entire side is to one of its parts, as the other entire side is to its corresponding part.*

**259.** By the transposition of the middle terms of a proportion, a new proportion may be formed; and thus in the figure under examination many more proportions may be proved to exist:

Thus, AB : AD = AC : AE
AB : AC = AD : AE
AB : BD = AC : EC
AB : AC = DB : EC
BD : DA = CE : EA
BD : CE = DA : EA

**260.** We will now seek the ratio which the base of the triangle bears to the parallel line intersecting the sides.

(Fig. 140.) Draw EG parallel to AB. From the preceding proposition we get the following proportion,

$$CB : CA = GB : EA$$

Now GB = ED, being opposite sides of a parallelogram,

therefore $CB : CA = ED : EA$

by transposition $CB : ED = CA : EA$

But $CA : EA = BA : DA$ (**258**)

therefore $CB : ED = BA : DA$

that is, *the base of a triangle is to the parallel line intercepted by the other sides, as one of those sides is to that part of it cut off by the parallel which lies next to the vertex of the triangle.*

**261.** *To find a fourth proportional line to three given lines.*

*Sol.* Let M, N, and O be the given lines, (fig. 141.) Draw two indefinite straight lines making any angle with each other. In one of them take AB = M, and BC = N. In the other take AD = O. Draw BD, and through C draw CE parallel to BD. DE is the line required.

For, BD is parallel to CE,

therefore $AB : BC = AD : DE$

that is, $M : N = O : DE$

**262.** *Remark* 1. If N and O are equal, the required line will be a third proportional to the lines M and N.

2. If M and N are equal, O will be equal to the required line.

**263.** *If the sides of an angle are divided proportionally, the straight lines joining the division points will be parallel.*

(Fig. 141.) If AC : AB = AE : AD
then BD is parallel to CE.

*Dem.* If BD is not parallel to CE, then a line may be drawn which shall be parallel; suppose it to be BX.

then AC : AB = AE : AX

But by supposition AC : AB = AE : AD

therefore AD = AX, which is impossible.

Therefore BD is parallel to CE.

*Conditions which determine the similarity of two Triangles.*

**264.** *Two triangles are similar, if two angles of the one are equal to two angles of the other, each to each.*

*Dem.* Let angle B = E, and C = F, (fig. 142.) Since two angles of one triangle are equal to two angles of the other, the third angle of the one must be equal to the third angle of the other, (**123.**) Consequently all the angles of the one are equal to the angles of the other, each to each.

To ascertain the proportionality of the sides, let us suppose the triangle ABC to be placed upon DEF, so that AB shall fall on DE, AC on DF, and BC on GH. Triangle ABC = DGH, (~~127~~;) consequently angle ABC = DGH, and angle ACB = DHG. But angle ABC = DEF, therefore DGH = DEF,) ~~112~~;) therefore GH is parallel to EF (112)

$$\begin{array}{l}\text{Therefore}\\ \text{or its equal}\end{array}\ \left.\begin{array}{c}\text{DG}\\ \text{AB}\end{array}\right\} : \text{DE} = \left.\begin{array}{c}\text{DH}\\ \text{or}\\ \text{AC}\end{array}\right\} : \text{DF}$$

$$\left.\begin{array}{c}\text{DG}\\ \text{or}\\ \text{AB}\end{array}\right\} : \text{DE} = \left.\begin{array}{c}\text{GH}\\ \text{or}\\ \text{BC}\end{array}\right\} : \text{EF}$$

$$\left.\begin{array}{c}\text{GH}\\ \text{or}\\ \text{BC}\end{array}\right\} : \text{EF} = \left.\begin{array}{c}\text{DH}\\ \text{or}\\ \text{AC}\end{array}\right\} : \text{DF}$$

Thus the homologous sides of the two triangles are proportional, and since the homologous angles are equal, the triangles are similar.

**265.** *Two triangles are similar, if the sides of one are proportional to the homologous sides of the other, each to each, and the included angles are equal.*

*Dem.* (Fig. 142.) Suppose AB : DE = AC : DF and angle A = D. Place the triangle ABC upon EDF so that point B shall fall on G, and point C on H. Then is triangle ABC = DGH, (**126**;) therefore BC = GH, angle ABC = DGH and ACB = DHG.

By supposition AB : DE = AC : DF

therefore DG : DE = DH : DF

therefore GH is parallel to EF; therefore angle DGH = DEF and angle DHG = DFE, (**108**;) consequently triangle DGH ∽ DEF; therefore triangle ABC ∽ DEF.

**266.** *Remark.* The sides which are opposite to equal angles are proportional; and the angles which are opposite to proportional sides are equal.

**267.** *Two triangles* ABC *and* DEF (fig. 143) *are similar, if the three sides of the one are proportional to the homologous sides of the other, each to each.*

*Dem.* Since the sides of the two triangles are by supposition proportional, we have to seek only for the equality of the homologous angles. It will be sufficient to demonstrate that onc angle of the one triangle is equal to the homologous angle of the other, for then, by the preceding proposition, the triangles will be similar.

Upon EF at the point E make an angle FEG = ABC; and at the point F make an angle EFG = ACB. Tri-

angles ABC and GEF are similar, (**264**;) therefore angle A = G. We have therefore the proportion;

$$AB : BC = GE : EF$$

$$\text{But} \quad AB : BC = DE : EF$$

$$\text{therefore} \quad GE : EF = DE : EF$$

therefore DE = GE.

$$\text{Again,} \quad AC : CB = GF : FE$$

$$\text{but} \quad AC : CB = DF : FE$$

$$\text{therefore} \quad GF : FE = DF : FE$$

therefore DF = GF.

Now EF = EF, therefore triangle GEF = DEF; therefore angle G = D. But angle G = A; therefore A = D. Consequently ABC ∽ DEF.

**268.** *Remark.* In the triangles ABC and DEF the proportional sides are opposite to the equal angles.

**269.** From the preceding propositions we learn that the similarity of two triangles may be inferred;

1. If two angles of the one are equal to the homologous angles of the other, each to each.
2. If two sides of the one are proportional to the homologous sides of the other, and the included angles equal.
3. If the three sides of the one are proportional to the homologous sides of the other.

We also learn that in similar triangles the equal angles are opposite to the proportional sides, and the proportional sides to the equal angles.

### 4. Miscellaneous Propositions.

**270.** *To construct upon the base* BC *a triangle which shall be similar to a given triangle*, DEF, (fig. 143.)

*Sol.* At the point B in the line BC make an angle ABC equal to FED, and at the point C make an angle ACB = EFD; then are triangles ABC and EFD similar, (**264.**)

**271.** *To determine the distance* AB, (fig. 144,) *which cannot be directly measured.*

*Sol.* Mark out a straight line AC, and take in it any point D. Measure angle A, and at D make the angle CDE = A. Produce DE until it meets a supposed line in the direction CB at the point E. We shall thus have triangle CDE. Since angle A = D therefore DE is parallel to AB; therefore CD : CA = DE : AB.

The three lines CD, CA, and DE may be measured; and therefore we can find AB. For example, if CD = $\frac{1}{4}$ CA, then DE = $\frac{1}{4}$ AB. Take the length of DE four times and we have AB.

**272.** *To prepare a scale of equal parts,* (fig. 145.)

This is a scale upon which straight lines or lengths are divided into a certain number of equal parts. It is used in drawing upon paper representations in a reduced size of lines and figures. Suppose it is required to draw a plan of a field having five sides; that is, to construct a pentagon similar to the field. It would not be convenient to construct upon paper a pentagon whose sides should be equal to those of the field; neither is it necessary so to do; we have only to make the angles upon paper equal to the corresponding angles in the field, and to draw the sides upon paper in the same ratio to each other as the sides of the field. Any length may be taken to represent a rod or a foot. Suppose one side of a field is 10 feet 9 inches in length, and that the diagram is to be made upon a scale of one foot to an inch;

that is, one inch on paper is to be taken as equal to one foot in the field. Measure between the points of the compasses a distance equal to 10 inches, and this will represent the 10 feet, then if the inches on the scale are divided into 12 equal parts, extend the opening of the compasses by a distance equal to 9 of these parts, for each of the parts represents one inch. Draw upon the paper an indefinite line, and take in it a part equal to the opening of the compasses. The use of a scale of equal parts is to enable us to construct small figures similar to great figures. To construct a scale of equal parts, some one unit of length, for example an inch, is taken as the basis, and several of these units are marked upon the rule, or the paper on which the scale is to be made. One of the extreme divisions is again divided into many equal parts. The unit assumed as the basis is generally small, and great accuracy is necessary in the division; if the division was made in the common mode, the marks would be so near one to another as not easily to be distinguished. A mode of division founded on the proportion of the sides of similar triangles has been adopted. In a straight line AB, (fig. 145,) take the parts AC, CD, DB, each equal to the unit of length assumed as the basis of the scale. Divide AC into 10 equal parts; then if $AC = 1$ inch, each of these parts will equal $\frac{1}{10}$ of an inch. Now to get the *hundredths* of an inch, construct upon AC any rectangular figure, for example the square AFMC. Divide each of the sides into 10 equal parts. Through the points of division of the lines AF and CM draw straight lines parallel to AC. Connect the division points of the lines AC and FM by diagonal lines; that is, connect 10 in one line with 9 in the other; 9 in one line with 8 in the other, &c. By the construction MN $= \frac{1}{10}$ AC,

therefore $MN : OP = MC : OC = 10 : 9$;
that is, $OP = \frac{9}{10}$ MN,

$$MN : QR = MC : QC = 10 : 8;$$

that is, $QR = \frac{8}{10}$ MN.

In like manner it may be demonstrated that UV has 7 of the parts of which MN has 10. Thus the extent MN is divided into 10 equal parts, and any one of these parts may be taken between the points of the compasses with the greatest ease. If $AC = 100$ feet, and we have a line 255 feet in length to be marked off; then extend the points of the compasses from 2 of the larger divisions to 5 on the 5th line of the subdivisions, that is on line GH.

Let us suppose a field ABCDE, (fig. 157,) is to be represented upon paper by a scale of 100 rods to an inch. We must first measure each of the sides and angles of the field. Suppose the side AB is 100 rods long; draw the line *a b* one inch long, and it will represent the side AB. At *b* make, with a protractor, an angle equal to angle B, and this will give us the direction of the next side *b c*, which, if BC is 120 rods in length, will be 1 inch and $\frac{2}{10}$ of an inch. At *c* make an angle equal to C, and we shall thus get the direction of the next side. Proceed in this manner till the construction of the figure is completed The figure *a b c d e* will be an accurate representation of the field; for, by construction, the angles of the one are equal to the homologous angles of the other; and the sides of the one are proportional to the homologous sides of the other. Therefore the two are similar figures.

**273.** *To find the perpendicular height* BC *of a tower. when it cannot be directly measured,* (fig. 146.)

*Sol.* 1. Measure upon the base of the tower the horizontal line AB. Draw upon paper the line ED representing AB upon a reduced scale. Measure the angle BAC, and make upon the paper the angle EDF = BAC. Erect at E the perpendicular EF. The triangle DEF is similar to ABC, (**264,**) because by construction angle EDF = BAC, and DEF = ABC;

therefore DE : EF = AB : BC.

But DE, EF, and AB are known quantities, therefore BC may be known. If DE = $\frac{3}{4}$ EF, then AB = $\frac{3}{4}$ BC and BC = $\frac{4}{3}$ AB; suppose AB = 60 feet; then BC = $\frac{4}{3}$ $\times$ 60 = 80 feet.

*Sol.* 2. Set up in the clear sunlight a perpendicular staff EF, (fig. 146.) Measure the length of the shadow ED which it casts. Measure also the length of the shadow which the tower makes at the same time. The rays of the sun's light may be considered as parallel; therefore, these rays, that is, the lines CA and FD make equal angles with the horizontal lines BA and ED; that is, angle CAB = FDE. Now angle CBA = FED, being right angles; therefore triangle CBA $\backsim$ FDE,

therefore DE : EF = AB : BC,

the height of the tower. That is, the length of the shadow of the staff is to the height of the staff as the length of the shadow of the tower is to the height of the tower.

**274.** *To find from the windows of a house the height of an object,* (fig. 147.)

Let A and B be the windows, one of which is perpendicularly over the other, and CG the height to be found.

*Sol.* Measure from A, the angle CAE, which an imaginary line CA drawn from the top of the object

makes with the imaginary horizontal line AE. Measure from B the angle CBD, made by the imaginary line CB with the imaginary horizontal line BD.

Now angle

$$CAB = CAE + EAB = CAE + 1\text{ R. A.} = CAE + 90^\circ;$$

and angle

$$ABC = ABD - CBD = 1\text{ R. A.} - CBD = 90^\circ - CBD.$$

Now we know the angles CAE and CBD, and therefore CAB and ABC may also be known; thus in the triangle ABC, the side AB and the two adjacent angles are known; therefore we can construct a similar triangle; and thus find the length of BC. We shall then know in the triangle BCD, the side BC and the angles CBD and CDB, (which is a right angle.) By means of these we can find the length of CD. Again, measure from B the angle DBG, made by DB with an imaginary line GB, drawn from the bottom of the object to the point B. In the triangle DBG the side DB and its adjacent angles are known; we can therefore find the length of DG. Add DG to CD and we shall have the whole height required.

**275.** *To find the height of an object which cannot be approached very nearly;* for example CD, (fig. 148.)

*Sol.* In the direction AD, measure the line AB. Measure also the angle CAB and CBD. By means of the scale of equal parts draw upon paper the line EF to represent AB. Make the angle FEG = CAB. Produce FE and make the angle GFH = CBD. From the point G let fall the perpendicular GH.

Triangle EFG ∽ ABC, therefore we have the proportions,

$$EF : FG = AB : BC,$$

which gives us the length of BC,

and FG : GH = BC : CD,

which gives us the required height.

**276.** *To find the distance* AB *between two inaccessible objects*, (fig. 149.)

*Sol.* Take a straight line CD, from each end of which both objects A and B can be seen. Measure the line CD, and also the angles ACD, BCD, ADC, BDC.

In the triangle ACD, the side CD, and the adjacent angles are known, we can therefore construct a similar triangle upon paper and thus find the length of DA.

In like manner we can find the length of DB, since we know the side CD and its adjacent angles in the triangle BCD.

Angle BDA = BDC — ADC, we know therefore two sides and the included angle of the triangle ADB. From these parts we can construct upon paper a triangle similar to ADB, and by comparing it with triangle ADB can find the length of AB; which was required.

**277.** *To divide a straight line into any number of equal parts.*

We have frequently had occasion to divide lines into equal parts, and have done so directly with the compasses. We will now do it by a method which admits of the accuracy of the division being demonstrated. Suppose it is required to divide the straight line AB into 5 equal parts, (fig. 150.)

*Sol.* Draw the indefinite line AM, making any angle with the given line AB. Take any distance AC, and measure it off 5 times upon the leg AM. We shall thus have AC = CD = DE = EF = FG. Join the last divi-

sion point G with B, and through the other division points C, D, E, and F, draw straight lines intersecting AB and parallel to GB. AB will be divided into 5 equal parts at the points of intersection H, I, K, and L.

*Dem.* Since CH is parallel to DI, we have the proportion,

$$AC : CD = AH : HI, \text{ (258,)}$$

$AC = DC$, therefore $AH = HI$. Again, DI is parallel to EK, therefore

$$AD : DE = AI : IK, \text{ (258.)}$$

Since $AD = 2\ DE$ therefore $AI = 2\ IK$. But $\frac{AI}{2} = HI = IK$; thus $AH = HI = IK$. By proceeding in a like manner, it may be demonstrated that $AH = HI = IK = KL = LB$.

**278.** *From the vertex of the right angle of the right-angled triangle* ABC, (fig. 151,) *let fall upon the hypothenuse the perpendicular* AD.

The following propositions can now be demonstrated.

1. *The two partial triangles* ABD *and* ADC *are similar to each other and to the whole triangle* ABC.

*Dem.* Angle $BAC = BDA$, being right angles; angle $B = B$, therefore triangle $ABC \backsim ABD$. It may be demonstrated in a like manner that triangle $ADC \backsim ABC$, therefore $ADC \backsim ADB$.

2. *Each of the sides* AB *and* AC *is a mean proportional between the whole hypothenuse and the portion of it, cut off by the perpendicular, which is adjacent to that side.*

*Dem.* In similar triangles the sides opposite to equal angles are proportional. In the triangles ABD and ABC, angle $BAD = ACB$; thus BD and BA are homologous sides, and since angle $BDA = BAC$, therefore

BA and BC are homologous sides. Hence we have this proportion,

BD : BA = BA : BC.

From the similarity of the triangles ADC and ABC we have the proportion,

CD : CA = CA : CB.

3. *The perpendicular* AD *is a mean proportional between the two parts into which the hypothenuse is divided.*

*Dem.* Triangle ADB ∽ ADC, therefore,

BD : AD = AD : DC;

which was to be demonstrated.

**279.** *To find a mean proportional between two given straight lines* M *and* N, (fig. 152.)

*Sol.* 1. In an indefinite straight line take AB = M and BC = N. Upon the whole line AC as a diameter describe a semi-circumference. At the point B erect a perpendicular intersecting the arc at D. BD is the mean proportional required.

*Dem.* Draw the chords AD and DC; angle ADC is a right angle, (**174**;) therefore we have this proportion, (**278**,)

AB or M : DB = DB : BC or N;

which was to be demonstrated.

*Sol.* 2. Take AC = M, and in this line take AB = N. Upon AC as a diameter describe a semi-circumference. At B erect a perpendicular which will intersect the arc at D. Draw AD, which will be the line required. For triangle ADC ∽ ADB, therefore

AC or M : AD = AD : AB or N;

which was to be demonstrated.

**280.** *Remark.* We have here solved another problem, namely, *to transform a rectangle into an equivalent*

*square.* For in the first solution we have $BD \times BD = \overline{BD}^2 = M \times N$, (**247**;) that is, the square constructed upon BD is equivalent to a rectangle of which M and N are adjacent sides. In the second solution we have $\overline{AD}^2 = M \times N$.

**281.** *What is the ratio of the perimeter of a triangle* ABC *to the perimeter of the similar triangle* DEF? (fig. 153.)

*Sol.* Triangle ABC $\backsim$ DEF, therefore

$$AB : BC = DE : EF.$$

In every proportion the sum of the 1st and 2d terms is to the 2d term as the sum of the 3d and 4th terms is to the 4th term; a new proportion is thus formed by *composition.*

Therefore $\quad AB + BC : BC = DE + EF : EF$

by transposition

$$AB + BC : DE + EF = BC : EF$$

But $\quad BC : EF = CA : FD$

therefore $\quad AB + BC : DE + EF = CA : FD$

by transposition $\quad AB + BC : CA = DE + EF : FD$

by composition

$$AB + BC + CA : CA = DE + EF + FD : FD$$

by transposition

$$AB + BC + CA : DE + EF + FD = CA : FD.$$

That is, *the perimeters of similar triangles are to one another as their homologous sides.*

**282.** *What is the ratio of the surface of the triangle* ABC *to that of the similar triangle* DEF?

*Sol.* Suppose triangle ABC to be placed upon DEF, so that its sides shall exactly coincide with those of the triangle GEH. Then triangle ABC would be equal to

GEH. We may therefore substitute GEH for ABC in our comparison.

Angle DFE $=$ GHE, therefore GH is parallel to DF, (**112.**) Draw GF; we thus have a third triangle GEF.

Triangle GEH : triangle GEF $=$ EH : EF (**250**)
triangle GEF : triangle DEF $=$ EG : ED (**250**)
But EG : ED $=$ EH : EF;
therefore
triangle GEF : triangle DEF $=$ EH : EF.

*Two proportions may be multiplied together, term by term, and a new proportion will thus be formed.*

From the preceding proportions we have

$$\text{GEH}\times\text{GEF} : \text{GEF}\times\text{DEF} = \text{EH}\times\text{EH} : \text{EF}\times\text{EF}.$$

The terms of either ratio may be divided by the same quantity, without destroying the proportion; hence we have

$$\text{GEH} : \text{DEF} = \overline{\text{EH}}^2 : \overline{\text{EF}}^2.$$

Triangle ABC $=$ GEH, and $\overline{\text{EH}}^2 = \overline{\text{BC}}^2$; therefore

$$\text{triangle ABC} : \text{triangle DEF} = \overline{\text{BC}}^2 : \overline{\text{EF}}^2;$$

that is, *the surfaces of similar triangles are to each other as the squares of their homologous sides.* For example, if BC $= \frac{1}{2}$ EF then triangle ABC $= \frac{1}{4}$ DEF.

If the homologous sides of two similar triangles are to one another

as 1 to 3, the surfaces will be as 1 to 9
2 " 7 " " 4 " 49, &c.

**283.** *The homologous sides of similar triangles are to one another as the square roots of the surfaces of the same triangles.*

If the surfaces of two similar triangles are to one another

as 1 : 4, the homologous sides will be as 1 : 2
1 : 9 " " " 1 : 3
36 : 100 " " " 6 : 10, &c.

### 5. POLYGONS IN GENERAL.

**284.** If the polygons ABCDE and *a b c d e*, (fig. 154,) are similar to each other, angle A = angle *a*, angle B = *b*, &c., &c.; the two polygons will be divided by homologous diagonals into similar triangles; that is, ABC $\sim$ *a b c*; ADC $\sim$ *a d c*, and ADE $\sim$ *a d e*.

*Dem.* Angle ABC = *a b c* and the sides AB and BC are proportional to the homologous sides *a b* and *b c*, therefore tringle ABC $\sim$ *a b c*.

Again, triangle ABC $\sim$ *a b c*, therefore angle ACB = *a c b*; angle BCD = *b c d*; therefore angle BCD — ACB = ACD = *b c d* — *a c b* = *a c d*. Since triangle ABC $\sim$ *a b c*,

therefore BC : *b c* = AC : *a c*
by supposition BC : *b c* = CD : *c d*
therefore AC : *a c* = CD : *c d*

consequently triangle ACD $\sim$ *a c d*. In a like manner it may be demonstrated that triangle AED $\sim$ *a e d*.

**285.** (Fig. 154.) If the triangle ABC $\sim$ *a b c*, ACD $\sim$ *a c d* and AED $\sim$ *a e d*, and these triangles are placed together in a similar order, so that the sides and angles of one set shall correspond in position with the equal angles and proportional sides of the other set, then the entire figure ABCDE will be similar to the entire figure *a b c d e*.

*Dem.* Triangle ABC $\sim$ *a b c*; ACD $\sim$ *a c d*; ADE $\sim$ *a d e*; consequently angle ABC = *a b c*; angle BCA = *b c a*; angle ACD = *a c d*; therefore angle BCA +

ACD = BCD = angle $bca + acd = bcd$. The equality of the other homologous angles in the two polygons can be demonstrated in a similar manner.

Since the triangles are similar, we have the following proportions among the sides;

$$
\begin{array}{rl}
 & AB : ab = BC : bc \\
 & BC : bc = CA : ca \\
 & CA : ca = CD : cd \\
\text{therefore} & BC : bc = CD : cd \\
\text{Again} & CD : cd = DA : da \\
 & DA : da = DE : de \\
\text{therefore} & CD : cd = DE : de \\
\text{Again} & DE : de = EA : ea \\
 & EA : ea = AD : ad \\
 & \phantom{EA : ea} = DC : dc \\
 & \phantom{EA : ea} = CA : ca \\
 & \phantom{EA : ea} = AB : ab
\end{array}
$$

Consequently in the two polygons not only are the homologous angles equal, but the homologous sides are proportional, therefore the polygons are similar.

**286.** (Fig. 155.) *To construct upon the line* FG *a figure which shall be similar to a given figure* ABCDE.

*Sol.* 1. Divide ABCDE by diagonals into 3 triangles; then beginning upon the line FG, construct 3 triangles similar to those of the polygon, and similarly disposed. For example, suppose FG to be homologous to AB, and upon it construct a triangle similar to ABC; then upon the side of this triangle which is homologous to side AC, construct a triangle similar to ADC; then upon the side of this second triangle which is homologous to side AD, construct a triangle similar to ADE,

and this will complete the figure, which will be similar to ABCDE.

*Sol.* 2. Produce AB to H, taking AH = FG. Produce the diagonals AC and AD, and the side AE. Draw HI parallel to BC, IK parallel to CD, KL parallel to DE. The required figure will be constructed. If FG is less than AB, so that the point H falls between the points A and B, the problem would be solved by drawing lines parallel to the sides of the given figure.

**287.** *To construct a diagram of a field upon a reduced scale.*

If the field can be measured in all directions from the inside, then divide it by diagonals into rectilineal triangles and measure all the sides of these triangles. Represent these triangles upon paper on a reduced scale in the same relative position that they occupy in the field, and thus the problem will be solved.

If the field cannot be measured from the inside, (as a piece of woodland,) but can be readily reached upon the outside, then measure all the sides of the perimeter, and all the angles. Draw the sides on a reduced scale upon paper, joining them together in their natural order, so as to form together the measured angles.

**288.** Suppose ABCDE $\backsim$ $a\,b\,c\,d\,e$, what is the ratio of AB + BC + CD + DE + EA to $a\,b + b\,c + c\,d + d\,e + e\,a$? (fig. 154.)

*Answer.*

$$AB : a\,b = BC : b\,c = CD : c\,d = DE : d\,e = EA : e\,a.$$

This is called a *continued proportion*, being a series of equal ratios.

*In every continued proportion the sum of all the ante-*

*cedents is to the sum of all the consequents as one antecedent is to its consequent.*

Therefore $AB + BC + CD + DE + EA : ab + bc + cd + de + ea = AB : ab = BC : bc$; that is, *the perimeters of similar figures are to each other as their homologous sides.*

**289.** *What ratio does the surface of* ABCDE *bear to that of* $abcde$? (Fig. 154.)

*Sol.* Divide the similar figures ABCDE and $abcde$ by homologous diagonals into similar triangles. We then have.

triangle ABC : triangle $abc = \overline{AC}^2 : \overline{ac}^2$ (**282**)

triangle ACD : triangle $acd = \overline{AC}^2 : \overline{ac}^2$

therefore,

triangle ABC : triangle $abc$ = triangle ACD : triangle $acd$.

Again,

triangle ACD : triangle $acd = \overline{AD}^2 : \overline{ad}^2$

triangle ADE : triangle $ade = \overline{AD}^2 : \overline{ad}^2$

therefore,

triangle ACD : triangle $acd$ = triangle ADE : triangle $ade$.

Because

triangle ABC : triangle $abc$ = triangle ACD : triangle $acd$ = triangle ADE : triangle $ade$

therefore,

triangle ABC + triangle ACD + triangle ADE : triangle $abc$ + triangle $acd$ + triangle $ade$ = triangle ABC : triangle $abc$ = triangle ADC : triangle $adc$ = triangle ADE : triangle $ade = \overline{AB}^2 : \overline{ab}^2 = \overline{BC}^2 : \overline{bc}^2$, &c., that is, *The surfaces of similar figures are to each other as the squares of their homologous sides.*

**290.** *Upon the* 3 *sides of a right-angled triangle construct similar figures, in which the sides of the triangle shall be similarly disposed, then the figure constructed upon the hypothenuse will be equal to the sum of the figures constructed upon the two sides which include the right angle.* Figure C = fig. A + fig. B, (fig. 156.)

*Dem.* Figure A : fig. B = side $\overline{a}^2$ : side $\overline{b}^2$,

therefore $\quad A + B : B = \overline{a}^2 + \overline{b}^2 : \overline{b}^2$

Again, $\quad B : C = \overline{b}^2 : \overline{c}^2$

therefore $\quad A + B : C = \overline{a}^2 + \overline{b}^2 : \overline{c}^2$;

$\overline{a}^2 + \overline{b}^2 = \overline{c}^2$, therefore A + B = C.

**291.** *To construct a figure which shall be similar to two given similar figures* ABCDE *and* $a\,b\,c\,d\,e$, *and shall be equivalent to both taken together*, (fig. 157.)

*Sol.* Make a right angle. Take one of the sides $f\,g$ equal to a side AB of one of the given figures, and the other side $f\,h$ equal to the homologous side, $a\,b$, of the other given figure. Draw the hypothenuse $h\,g$, and upon this line $h\,g$ construct a figure similar to ABCDE and $a\,b\,c\,d\,e$, and in which the side $h\,g$ shall be homologous to the sides AB and $a\,b$. It has been demonstrated in the preceding proposition that the figure constructed upon $h\,g$ will be equivalent to both the figures constructed upon the sides $f\,h$ and $f\,g$, or their equals AB and $a\,b$, taken together.

### 5. Circles.

**292.** *If two chords cut each other in a circle, the parts are reciprocally proportional*, (fig. 158.)

The part AE of the chord AB is to the part CE of the chord CD, not as the second part EB of the first

chord is to the second part ED of the other chord, but reciprocally as ED : EB; that is, AE : CE = ED : EB.

*Dem.* Draw DB and AC. We have angle AEC = BED; angle ACE = EBD; angle CAE = EDB; therefore triangle AEC ∾ triangle DEB; therefore

AE : EC = ED : EB,

for they are homologous sides of similar triangles

**293.** *If two chords being produced cut each other, then the entire lines will be reciprocally proportional to the parts without the circle,* (fig. 159.)

OA : OC = OD : OB.

*Dem.* Draw AD and BC. Angle AOD = angle COB; angle OAD = angle OCB; therefore triangle OAD ∾ triangle OCB; therefore

OA : OC = OD : OB.

The entire lines AO and CO are called *secants.*

**294.** *If the secant* AC *meets the tangent* CD *of the same circle, then is the tangent* CD *a mean proportional between the entire line* AC *and the part without the circle,* (fig. 160.)

*Dem.* Draw DB and DA. Angle DCB = angle DCA. The angle CDB = CAD, (**178**,) therefore the triangles CDB and CAD are similar. Consequently the homologous sides are proportional; that is,

CA : CD = CD : CB.

**295.** *Ratio of the circumferences and of the surfaces of two circles.*

Every circle can be considered as a regular polygon with an infinite number of sides. Two circles can therefore be considered as regular polygons having an equal number of sides. Regular polygons having an

equal number of sides are similar, and therefore the circles can be treated as similar figures. The circumferences and surfaces of circles will be to one another as the perimeters and surfaces of similar polygons are to one another, viz., as their homologous sides and the squares of those sides. The homologous sides of two circles are two chords subtending equal angles, by joining the extremities of which to the centre of the circle similar triangles will be formed. Therefore the two small chords will be to each other as the radii of their respective circles. Consequently *the circumferences of two circles are to one another as their radii or diameters; and their surfaces as the squares of the radii or of the diameters.*

If the radii of two circles are to each other as

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 to 2 | the circum'nces | will be as | 1 : 2 | the surfaces as | | 1 : 4 |
| 1 " 3 | " | " | 1 : 3 | " | | 1 : 9 |
| 1 " 4 | " | " | 1 : 4 | " | | 1 : 16 |
| 2 " 3 | " | " | 2 : 3 | " | | 4 : 9 |

Conversely, *the diameters and radii of two circles are to one another as the square roots of the surfaces.*

**296.** *To describe a circle the surface of which shall be equivalent to the surfaces of two given circles taken together.*

Make a right angle; let one of the sides be equal to the diameter of one circle, and the other side be equal to the diameter of the other circle; draw the hypothenuse, and upon it as a diameter describe a circle, and this circle will be equivalent to the two given circles taken together. For all circles are similar figures, consequently the circle, whose diameter is the hypothenuse of a right-angled triangle, will be equivalent to both the circles whose diam-

eters are the sides which enclose the right angle taken togther.

### 6. Solids.

**297.** *Ratio of two cubes.*

The solidity of a cube is the product of three equal factors, each of which is the measure of a side of the cube; therefore the *solidities of two cubes* will be to each other in the ratio of these products; that is, *they will be to each other as the cubes of their sides.* If the side of one cube is to the side of another cube as

| | | | | |
|---|---|---|---|---|
| 1 : 2 | then their solidities will be as | | | 1 : 8 |
| 1 : 3 | " | " | " | 1 : 27 |
| 1 : 4 | " | " | " | 1 : 64 |
| 3 : 4 | " | " | " | 27 : 64 |

Conversely, *the sides of two cubes are to each other as the cube roots of their solidities.*

**298.** *Ratio of two cylinders or prisms.*

The solidity of a cylinder is the product of its base multiplied by its altitude. Consequently the solidities of two cylinders are to one another in the ratio of these products. If the altitudes of two cylinders be equal, the cylinders are to each other as their bases. If the bases be equal, they are to each other as their altitudes. If the solidities of two cylinders are equal, their bases are to each other reciprocally as their altitudes.

**299.** *Ratio of the superficial and of the solid contents of two spheres.*

Since the surface of every sphere is equal to 4 times the surface of a great circle of that sphere, therefore the surfaces of two spheres are to each other in the ratio of 4 times the surfaces of their great circles, or in the ratio

of their great circles, and consequently in the ratio of the squares of their radii, or of their diameters.

From what has been said before, it follows (**208**,) that the solidity of a sphere whose diameter is D, is equal to the product of $D \times D \times D \times .523$. The solidity of a sphere whose diameter is $d$ is equal to the product of $d \times d \times d \times .523$. Consequently, the solidity of the two spheres will be to each other as $D \times D \times D \times .523 : d \times d \times d \times .523$; that is, *the solidities of two spheres are to each other as the cubes of the diameters of the spheres*, and, consequently, as the cubes of the radii.

If the diameter of one sphere is

| | | | | |
|---|---|---|---|---|
| 1 | and of another | 2 | their solidities are as | 1 : 8 |
| 1 | " | 3 | " | 1 : 27 |
| 5 | " | 6 | " | 125 : 216 |

Conversely, *the diameters of two spheres are to each other as the cube roots of the solidities of the same spheres.* It would be a great mistake, therefore, to infer that the surface or the diameter of the sun is 1,400,000 as great as that of the earth, because the sun is 1,400,000 as great as the earth.

**300.** *Ratio of the solidity of a sphere to the solidity of a circumscribed cube;* that is, a cube the sides of which are each equal to the diameter of the sphere. It is evident that the solidity of such a cube would be greater than the solidity of the sphere, since the faces of the cube would be tangents to the sphere, and thus the sphere would be contained in the cube. What ratio will the solidity of the one body bear to that of the other?

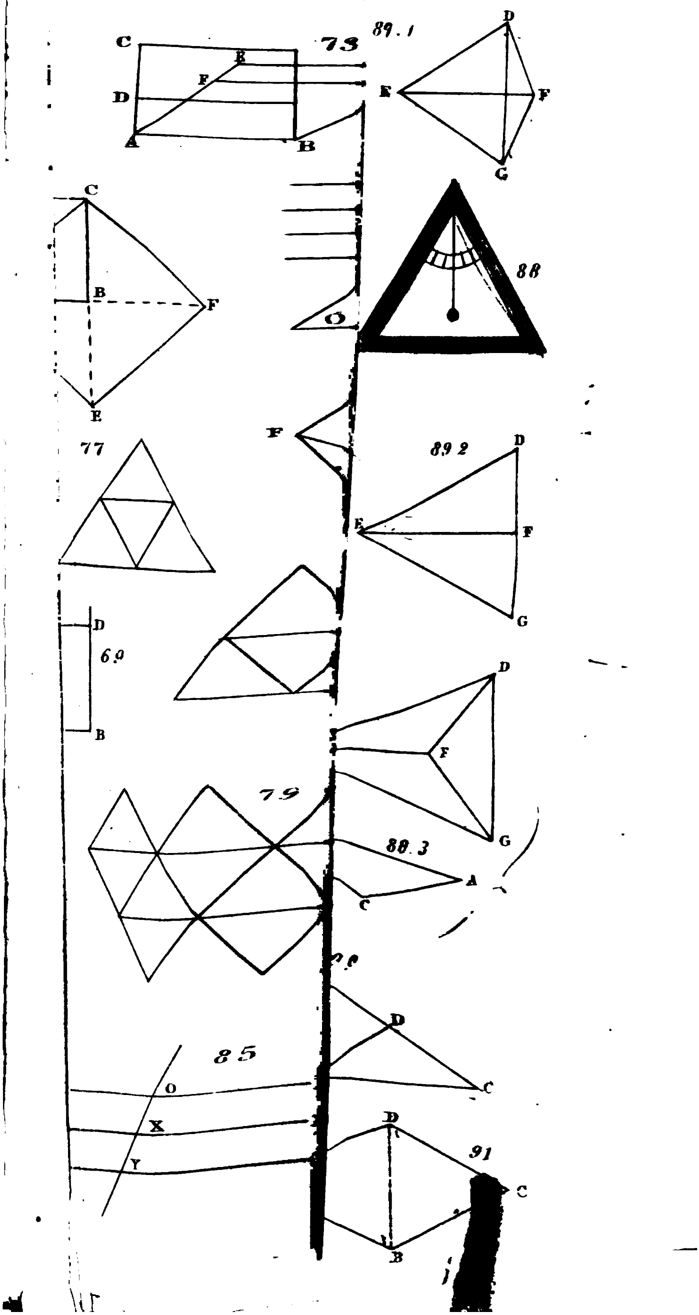
*Sol.* Let us call the diameter of the sphere $d$, then will the solidity of the sphere $= \frac{314}{100} \times \frac{ddd}{6}$, and the solidity of the cube $= ddd$; consequently the solidity

of the sphere will be to the solidity of the cube of its diameter as $\frac{314}{100} \times \frac{ddd}{6} : ddd = 314 : 600 = 157 : 300.$

**301.** *Ratio of the solidity of a sphere to the solidity of a circumscribed cylinder;* that is, a cylinder whose altitude is equal to the diameter of the sphere, and whose bases are equal to great circles of the sphere.

Call the diameter of the sphere *d;* then will its solidity $= \frac{314}{100} \times \frac{ddd}{6}$; the solidity of the cylinder $= \frac{314}{100} \times \frac{ddd}{4}$; consequently the solidity of the sphere is to the solidity of the circumscribed cylinder, as

$$\frac{314}{100} \times \frac{ddd}{6} : \frac{314}{100} \times \frac{ddd}{4} = 4 : 6 = 2 : 3.$$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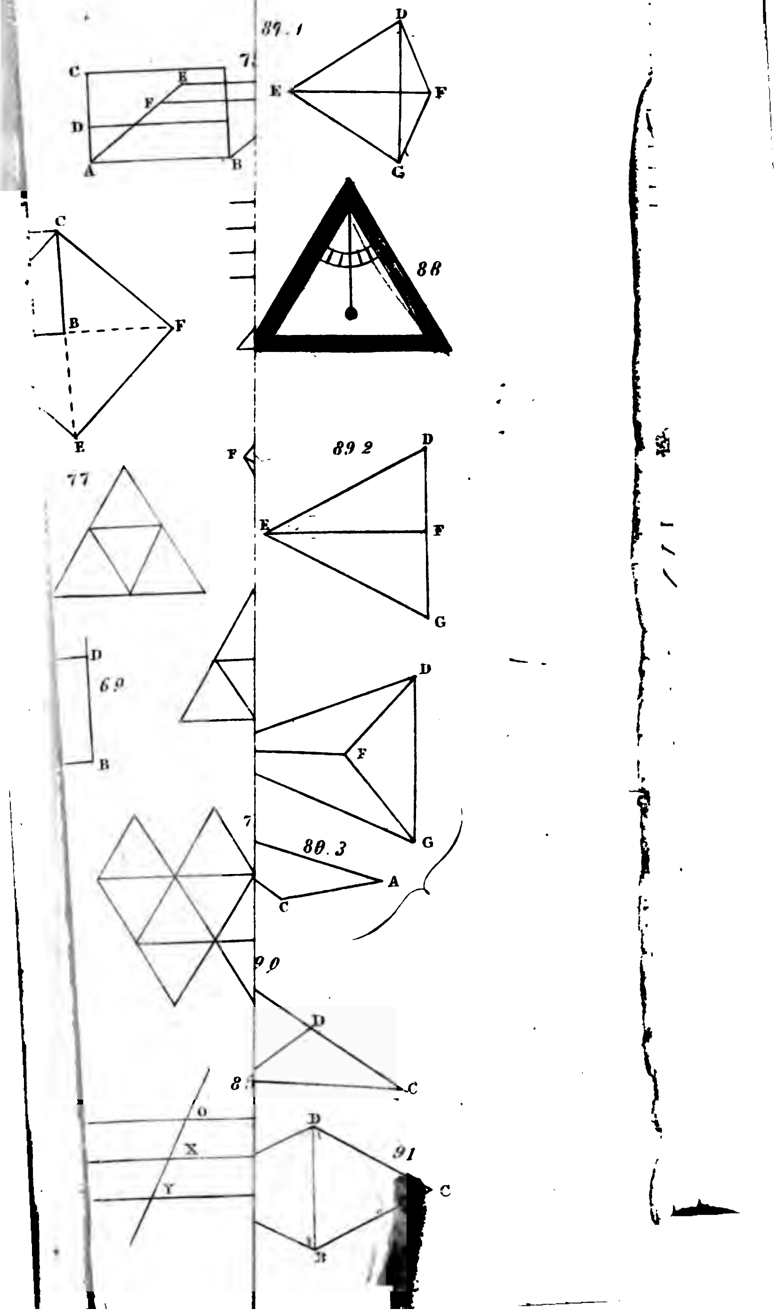

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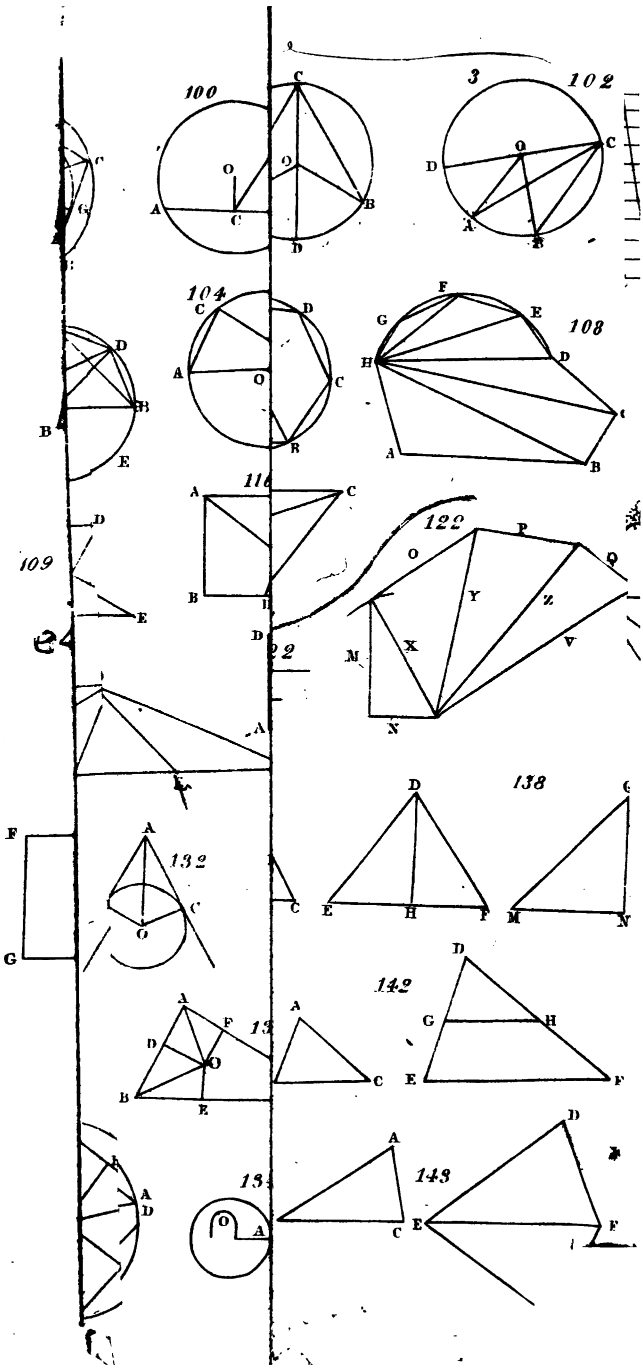
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